Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms

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Abstract

This is the sixth part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. In this paper (Part VI), we construct the appropriate natural associativity isomorphisms between triple tensor product functors. In fact, we establish a "logarithmic operator product expansion" theorem for logarithmic intertwining operators. In this part, a great deal of analytic reasoning is needed; the statements of the main theorems themselves involve convergence assertions.

Contents

In this paper, Part VI of a series of eight papers on logarithmic tensor category theory, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: In Part I [HLZ1], which contains Sections 1 and 2, we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. In Part II [HLZ2], which contains Section 3, we develop logarithmic formal calculus and study logarithmic intertwining operators. In Part III [HLZ3], which contains Section 4, we introduce and study intertwining maps and tensor product bifunctors. In Part IV [HLZ4], which contains Sections 5 and 6, we give constructions of the $P(z)$- and

$Q(z)$-tensor product bifunctors using what we call "compatibility conditions" and certain other conditions. In Part V [HLZ5], which contains Sections 7 and 8, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. The present paper, Part VI, contains Sections 9 and 10. In Part VII [HLZ6], which contains Section 11, we give sufficient conditions for the existence of the associativity isomorphisms. In Part VIII [HLZ7], which contains Section 12, we construct braided tensor category structure.

9 The expansion condition for intertwining maps and the associativity of logarithmic intertwining operators

In the present section, we establish results, especially Theorem 9.17, that form the crucial technical foundation of the construction of the natural associativity isomorphisms in Section 10. In Section 7 we have studied the conditions necessary for products and iterates of certain intertwining maps to exist. In Section 8 we have proved that products and iterates of such intertwining maps give elements of the dual space of the vector space tensor product of the three objects involved that satisfy the $P(z_1, z_2)$-compatibility condition and the $P(z_1, z_2)$-local grading restriction condition. In Theorem 5.50, we have given a characterization of $W_1 \boxtimes_{P(z)} W_2$ in terms of the $P(z)$-compatibility condition and the $P(z)$-local grading restriction condition. It is natural to also try to characterize the subspaces of the dual space mentioned above by means of the $P(z_1, z_2)$-compatibility condition and the $P(z_1, z_2)$-local grading restriction condition, but this time, these two conditions are not enough. We have to find additional conditions such that the elements satisfying all of the appropriate conditions can be obtained from both products and iterates of suitable intertwining maps. These additional conditions constitute what we will call the "expansion condition."

Assuming that the convergence condition in Section 7 is satisfied, in this section we study the conditions for the products of suitable intertwining maps to be expressible as the iterates of some suitable intertwining maps, and vice versa. To do this, following the idea in [H], we first study certain properties satisfied by the product, and separately, the iterate, of two intertwining maps. Then we obtain the deeper result that if a product satisfies the properties naturally satisfied by iterates, it can indeed be expressed as an iterate, and vice versa for an iterate. Using these results, we prove near the end of this section that the condition that products satisfy the properties for iterates and the condition that iterates satisfy the properties for products are equivalent to each other, and we introduce the term "expansion condition for intertwining maps" for either of these equivalent conditions (Definition 9.28).

We show that under the assumption of the convergence condition and the expansion condition, along with certain "minor" conditions, a product or iterate of two intertwining maps can be expressed as an iterate or product, respectively. Using the correspondence between intertwining maps and logarithmic intertwining operators, these results give the "associativity of logarithmic intertwining operators," which says that a product of logarithmic intertwining operators can be expressed as an iterate of logarithmic intertwining operators. This associativity of logarithmic intertwining operators is in fact a strong version of the "logarithmic operator product expansion," which in turn is the starting point of "logarithmic conformal field theory," studied extensively by physicists and mathematicians, as we have discussed in the Introduction. In this section, this logarithmic operator product expansion is established as a mathematical theorem. These results are also viewed as saying that products or iterates of intertwining maps or of logarithmic intertwining operators uniquely "factor through" suitable tensor product generalized modules.

The results in the present section are generalizations to the logarithmic setting of the corresponding results in the finitely reductive case obtained in [H]. See Remark 9.18 for a comparison of the main results in the present section with the corresponding results in [H], including the corrections of some minor mistakes. The most crucial results are Theorem 9.17, which essentially constructs the intermediate generalized module, and Lemma 9.22, which essentially constructs the corresponding intertwining map. The main difficult aspect of the proofs of these results is that we have to prove that certain iterated series converge and that their sums are equal to the sums of iterated series obtained by changing the order of summation. These difficulties, embedding in "formal calculus," are overcome either by proving the absolute convergence of the associated double or triple series or by using the convergence of suitable Taylor expansions. The reasoning requires considerable use of analytic methods.

We again recall our Assumptions 4.1, 5.30 and 7.11 concerning our category $\mathcal{C}$.

In this section z_1 and z_2 will be distinct nonzero complex numbers, and

$$z_0 = z_1 - z_2;$$

we shall make various assumptions on these numbers below.

For objects W_1, W_2, W_3, W_4, M_1 and M_2 of $\mathcal{C}$, let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Then under the assumption of the convergence condition for intertwining maps in $\mathcal{C}$ (recall Proposition 7.3 and Definition 7.4), when $|z_1| > |z_2| > |z_0| > 0$, both the product $I_1 \circ (1_{W_1} \otimes I_2)$ and the iterate $I^1 \circ (I^2 \otimes 1_{W_3})$ exist and are $P(z_1, z_2)$-intertwining maps, by Proposition 8.5. In this section we will consider when such a product can be expressed as such an iterate and vice versa. To compare these two types of maps, we shall study some conditions specific to each type.

Here and below, we let W_1, W_2 and W_3 be arbitrary generalized V-modules. (Later, these modules will be assumed to be objects of $\mathcal{C}$.) We start with the following:

Definition 9.1 Let
$$\lambda \in (W_1 \otimes W_2 \otimes W_3)^*.$$

For $w_{(1)} \in W_1$, we define the *evaluation of λ at $w_{(1)}$* to be the element

$$\mu^{(1)}_{\lambda, w_{(1)}} \in (W_2 \otimes W_3)^*$$

given by

$$\mu^{(1)}_{\lambda, w_{(1)}}(w_{(2)} \otimes w_{(3)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

for $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. For $w_{(3)} \in W_3$, we define the *evaluation of λ at $w_{(3)}$* to be the element

$$\mu^{(2)}_{\lambda, w_{(3)}} \in (W_1 \otimes W_2)^*$$

given by

$$\mu^{(2)}_{\lambda, w_{(3)}}(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

Remark 9.2 Given $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$, $w_{(1)} \in W_1$ and $w_{(3)} \in W_3$, it is natural to ask whether the evaluations $\mu^{(1)}_{\lambda, w_{(1)}} \in (W_2 \otimes W_3)^*$ and $\mu^{(2)}_{\lambda, w_{(3)}} \in (W_1 \otimes W_2)^*$ of λ satisfy the $P(z)$-compatibility condition (recall (5.141)) for some suitable nonzero complex numbers z, under suitable conditions. In fact, the formal computations underlying the next lemma suggest that when λ satisfies the $P(z_1, z_2)$-compatibility condition (recall (8.44)), these evaluations of λ "almost" satisfy the $P(z_2)$-compatibility condition and the $P(z_0)$-compatibility condition, respectively. However, even when λ does satisfy the $P(z_1, z_2)$-compatibility condition, in general these evaluations of λ do not even satisfy the $P(z_2)$-lower truncation condition (Part (a) of the $P(z_2)$-compatibility condition) or the $P(z_0)$-lower truncation condition (Part (a) of the $P(z_0)$-compatibility condition). In particular, when λ in (5.141) is replaced by $\mu^{(1)}_{\lambda, w_{(1)}}$ and $z = z_2$, the right-hand side of (5.141) might not exist in the usual algebraic sense, and similarly, when λ in (5.141) is replaced by $\mu^{(2)}_{\lambda, w_{(3)}}$ and $z = z_0$, the right-hand side of (5.141) might not exist algebraically, and for this reason $\mu^{(1)}_{\lambda, w_{(1)}}$ and $\mu^{(2)}_{\lambda, w_{(3)}}$ do not in general satisfy the $P(z_2)$-compatibility condition or the $P(z_0)$-compatibility condition. But the next result, which generalizes Lemma 14.3 in [H], asserts that if λ satisfies the $P(z_1, z_2)$-compatibility condition, then in both cases, under the natural assumptions on the complex numbers, the right-hand side of (5.141) exists *analytically* and (5.141) holds *analytically*, in the sense of weak absolute convergence, as discussed in Remark 7.24.

Lemma 9.3 *Assume that $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ satisfies the $P(z_1, z_2)$-compatibility condition (recall (8.44)). If $|z_2| > |z_0| \; (> 0)$, then for any $v \in V$ and $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$,*

$$\left(Y'_{P(z_0)}(v, x) \mu^{(2)}_{\lambda, w_{(3)}} \right)(w_{(1)} \otimes w_{(2)})$$

$$= \operatorname{Res}_{x_0^{-1}} \left(\tau_{P(z_1, z_2)} \left(x \delta \left(\frac{x_0^{-1} - z_2}{x^{-1}} \right) \right. \right.$$

$$\cdot Y_t\big((-x_0^2)^{L(0)}e^{-x_0 L(1)}e^{xL(1)}(-x^{-2})^{L(0)}v,x_0\big)\Big)\lambda\bigg)(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})$$

$$-\operatorname{Res}_{x_0^{-1}}x\delta\Big(\frac{-z_2+x_0^{-1}}{x^{-1}}\Big)\cdot$$
$$\cdot\lambda(w_{(1)}\otimes w_{(2)}\otimes Y_3(e^{xL(1)}(-x^{-2})^{L(0)}v,x_0^{-1})w_{(3)}),\tag{9.1}$$

the coefficients of the monomials in x and x_1 in

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\Big(Y'_{P(z_0)}(v,x)\mu^{(2)}_{\lambda,w_{(3)}}\Big)(w_{(1)}\otimes w_{(2)})$$

are absolutely convergent, and we have

$$\Big(\tau_{P(z_0)}\Big(x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)Y_t(v,x)\Big)\mu^{(2)}_{\lambda,w_{(3)}}\Big)(w_{(1)}\otimes w_{(2)})$$
$$=x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\Big(Y'_{P(z_0)}(v,x)\mu^{(2)}_{\lambda,w_{(3)}}\Big)(w_{(1)}\otimes w_{(2)}).\tag{9.2}$$

Analogously, if $|z_1|>|z_2|\ (>0)$, then for any $v\in V$ and any $w_{(j)}\in W_j$,

$$\Big(Y'_{P(z_2)}(v,x)\mu^{(1)}_{\lambda,w_{(1)}}\Big)(w_{(2)}\otimes w_{(3)})$$
$$=(Y'_{P(z_1,z_2)}(v,x_0)\lambda)(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})$$
$$-\operatorname{Res}_{x_1}x_0\delta\Big(\frac{z_1+x_1}{x_0^{-1}}\Big)\lambda(Y_1((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_1)w_{(1)}\otimes w_{(2)}\otimes w_{(3)}),\tag{9.3}$$

the coefficients of the monomials in x and x_1 in

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_2}{x_1}\Big)\Big(Y'_{P(z_2)}(v,x)\mu^{(1)}_{\lambda,w_{(1)}}\Big)(w_{(2)}\otimes w_{(3)})$$

are absolutely convergent, and we have

$$\Big(\tau_{P(z_2)}\Big(x_1^{-1}\delta\Big(\frac{x^{-1}-z_2}{x_1}\Big)Y_t(v,x)\Big)\mu^{(1)}_{\lambda,w_{(1)}}\Big)(w_{(2)}\otimes w_{(3)})$$
$$=x_1^{-1}\delta\Big(\frac{x^{-1}-z_2}{x_1}\Big)\Big(Y'_{P(z_2)}(v,x)\mu^{(1)}_{\lambda,w_{(1)}}\Big)(w_{(2)}\otimes w_{(3)}).\tag{9.4}$$

Proof First, for our distinct nonzero complex numbers z_1 and z_2 with $z_0=z_1-z_2$, by definition of the action $\tau_{P(z_1,z_2)}$ (8.29) we have

$$x_0\delta\Big(\frac{z_1+x_1}{x_0^{-1}}\Big)x_2^{-1}\delta\Big(\frac{z_0+x_1}{x_2}\Big)\lambda(Y_1((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_1)w_{(1)}\otimes w_{(2)}\otimes w_{(3)})$$
$$+x_0\delta\Big(\frac{z_2+x_2}{x_0^{-1}}\Big)x_1^{-1}\delta\Big(\frac{-z_0+x_2}{x_1}\Big)\lambda(w_{(1)}\otimes Y_2((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_2)w_{(2)}\otimes w_{(3)})$$
$$=\Big(\tau_{P(z_1,z_2)}\Big(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)\lambda\Big)(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})$$
$$-x_1^{-1}\delta\Big(\frac{-z_1+x_0^{-1}}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-z_2+x_0^{-1}}{x_2}\Big)\lambda(w_{(1)}\otimes w_{(2)}\otimes Y_3^o(v,x_0)w_{(3)})\tag{9.5}$$

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for any $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Replacing v by

$$(-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v,$$

using formula (5.3.1) in [FHL], and then taking $\mathrm{Res}_{x_0^{-1}}$ we get:

$$
\begin{aligned}
& x_2^{-1} \delta\Big(\frac{z_0 + x_1}{x_2}\Big) \lambda(Y_1(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& + x_1^{-1} \delta\Big(\frac{-z_0 + x_2}{x_1}\Big) \lambda(w_{(1)} \otimes Y_2(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_2) w_{(2)} \otimes w_{(3)}) \\
& = \mathrm{Res}_{x_0^{-1}} \Big(\tau_{P(z_1,z_2)} \Big(x_1^{-1} \delta\Big(\frac{x_0^{-1} - z_1}{x_1}\Big) x_2^{-1} \delta\Big(\frac{x_0^{-1} - z_2}{x_2}\Big) \cdot \\
& \quad \cdot Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0) \Big) \lambda \Big) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)})) \\
& - \mathrm{Res}_{x_0^{-1}} x_1^{-1} \delta\Big(\frac{-z_1 + x_0^{-1}}{x_1}\Big) x_2^{-1} \delta\Big(\frac{-z_2 + x_0^{-1}}{x_2}\Big) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0^{-1}) w_{(3)}). \qquad (9.6)
\end{aligned}
$$

By (5.86), the left-hand side of (9.6) is equal to

$$\Big(\tau_{P(z_0)} \Big(x_1^{-1} \delta\Big(\frac{x_2 - z_0}{x_1}\Big) Y_t(v, x_2^{-1}) \Big) \mu^{(2)}_{\lambda, w_{(3)}} \Big) (w_{(1)} \otimes w_{(2)}). \qquad (9.7)$$

Taking Res_{x_1} gives

$$(\tau_{P(z_0)}(Y_t(v, x_2^{-1})) \mu^{(2)}_{\lambda, w_{(3)}})(w_{(1)} \otimes w_{(2)}) = (Y'_{P(z_0)}(v, x_2^{-1}) \mu^{(2)}_{\lambda, w_{(3)}})(w_{(1)} \otimes w_{(2)}). \qquad (9.8)$$

By the $P(z_1, z_2)$-compatibility condition and formula (8.51) in Remark 8.18, the first term on the right-hand side of (9.6) equals

$$
\begin{aligned}
& x_1^{-1} \delta\Big(\frac{x_2 - z_0}{x_1}\Big) \mathrm{Res}_{x_0^{-1}} \Big(\tau_{P(z_1,z_2)} \Big(x_2^{-1} \delta\Big(\frac{x_0^{-1} - z_2}{x_2}\Big) \cdot \\
& \quad \cdot Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0) \Big) \lambda \Big) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)})). \qquad (9.9)
\end{aligned}
$$

Now suppose that $|z_2| > |z_0|$ (> 0). Then formula (8.7) holds. From this and (9.9), the right-hand side of (9.6) becomes

$$
\begin{aligned}
& x_1^{-1} \delta\Big(\frac{x_2 - z_0}{x_1}\Big) \mathrm{Res}_{x_0^{-1}} \Big(\tau_{P(z_1,z_2)} \Big(x_2^{-1} \delta\Big(\frac{x_0^{-1} - z_2}{x_2}\Big) \cdot \\
& \quad \cdot Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0) \Big) \lambda \Big) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& - x_1^{-1} \delta\Big(\frac{x_2 - z_0}{x_1}\Big) \mathrm{Res}_{x_0^{-1}} x_2^{-1} \delta\Big(\frac{-z_2 + x_0^{-1}}{x_2}\Big) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0^{-1}) w_{(3)}).
\end{aligned}
$$

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By taking Res_{x_1}, we erase the two factors $x_1^{-1}\delta\left(\frac{x_2 - z_0}{x_1}\right)$, leaving an expression which is exactly the right-hand side of (9.1) with x replaced by x_2^{-1} (thus proving (9.1)) and, while not lower truncated in x_2^{-1}, can still be multiplied by $x_1^{-1}\delta\left(\frac{x_2 - z_0}{x_1}\right)$ (when $|z_2| > |z_0| > 0$), in the sense of absolute convergence, yielding this expression again. That is, let X be either side of (9.6). Then

$$X = x_1^{-1}\delta\left(\frac{x_2 - z_0}{x_1}\right)\mathrm{Res}_{x_1} X,$$

in this sense of convergence. Applying this to (9.7) and (9.8) gives

$$\left(\tau_{P(z_0)}\left(x_1^{-1}\delta\left(\frac{x_2 - z_0}{x_1}\right)Y_t(v, x_2^{-1})\right)\mu_{\lambda, w_{(3)}}^{(2)}\right)(w_{(1)} \otimes w_{(2)})$$
$$= x_1^{-1}\delta\left(\frac{x_2 - z_0}{x_1}\right)\left(Y'_{P(z_0)}(v, x_2^{-1})\mu_{\lambda, w_{(3)}}^{(2)}\right)(w_{(1)} \otimes w_{(2)}), \tag{9.10}$$

proving (9.2) (with x in (9.2) replaced by x_2^{-1}).

Analogously, for our distinct nonzero complex numbers z_1 and z_2 with $z_0 = z_1 - z_2$, we can also write the definition of $\tau_{P(z_1, z_2)}$ as

$$x_0\delta\left(\frac{z_2 + x_2}{x_0^{-1}}\right)x_1^{-1}\delta\left(\frac{-z_0 + x_2}{x_1}\right)\lambda(w_{(1)} \otimes Y_2((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v, x_2)w_{(2)} \otimes w_{(3)})$$
$$+x_1^{-1}\delta\left(\frac{-z_1 + x_0^{-1}}{x_1}\right)x_2^{-1}\delta\left(\frac{-z_2 + x_0^{-1}}{x_2}\right)\lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3^o(v, x_0)w_{(3)})$$
$$= \left(\tau_{P(z_1, z_2)}\left(x_1^{-1}\delta\left(\frac{x_0^{-1} - z_1}{x_1}\right)x_2^{-1}\delta\left(\frac{x_0^{-1} - z_2}{x_2}\right)Y_t(v, x_0)\right)\lambda\right)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$
$$-x_0\delta\left(\frac{z_1 + x_1}{x_0^{-1}}\right)x_2^{-1}\delta\left(\frac{z_0 + x_1}{x_2}\right)\lambda(Y_1((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v, x_1)w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$
$$\tag{9.11}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Taking Res_{x_1} we get

$$x_0\delta\left(\frac{z_2 + x_2}{x_0^{-1}}\right)\lambda(w_{(1)} \otimes Y_2((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v, x_2)w_{(2)} \otimes w_{(3)})$$
$$+x_2^{-1}\delta\left(\frac{-z_2 + x_0^{-1}}{x_2}\right)\lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3^o(v, x_0)w_{(3)})$$
$$= \left(\tau_{P(z_1, z_2)}\left(x_2^{-1}\delta\left(\frac{x_0^{-1} - z_2}{x_2}\right)Y_t(v, x_0)\right)\lambda\right)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$
$$-\mathrm{Res}_{x_1}x_0\delta\left(\frac{z_1 + x_1}{x_0^{-1}}\right)x_2^{-1}\delta\left(\frac{z_0 + x_1}{x_2}\right)\cdot$$
$$\cdot\lambda(Y_1((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v, x_1)w_{(1)} \otimes w_{(2)} \otimes w_{(3)}). \tag{9.12}$$

By the definition of $\tau_{P(z_2)}$ (5.86) and formula (5.3.1) in [FHL], the left-hand side of (9.12) is equal to

$$\left(\tau_{P(z_2)}\left(x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)Y_t(v,x_0)\right)\mu_{\lambda,w_{(1)}}^{(1)}\right)(w_{(2)}\otimes w_{(3)}),$$

and taking Res_{x_2} gives

$$(\tau_{P(z_2)}(Y_t(v,x_0))\mu_{\lambda,w_{(1)}}^{(1)})(w_{(2)}\otimes w_{(3)}) = (Y'_{P(z_2)}(v,x_0)\mu_{\lambda,w_{(1)}}^{(1)})(w_{(2)}\otimes w_{(3)}).$$

Now suppose that $|z_1| > |z_2| > 0$. Then (8.4) holds, and by (8.50), which follows from the $P(z_1,z_2)$-compatibility condition, the right-hand side of (9.12) becomes

$$x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)(Y'_{P(z_1,z_2)}(v,x_0)\lambda)(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})$$

$$-x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)\mathrm{Res}_{x_1}x_0\delta\left(\frac{z_1+x_1}{x_0^{-1}}\right)\cdot$$

$$\cdot\lambda(Y_1((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_1)w_{(1)}\otimes w_{(2)}\otimes w_{(3)}).$$

Just as in the proof of (9.2), we take Res_{x_2} to obtain the right-hand side of (9.3) (thus proving (9.3)) and then multiply by $x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)$, yielding the same expression, and we obtain

$$\left(\tau_{P(z_2)}\left(x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)Y_t(v,x_0)\right)\mu_{\lambda,w_{(1)}}^{(1)}\right)(w_{(2)}\otimes w_{(3)})$$

$$= x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)\left(Y'_{P(z_2)}(v,x_0)\mu_{\lambda,w_{(1)}}^{(1)}\right)(w_{(2)}\otimes w_{(3)}), \qquad (9.13)$$

proving (9.4). $\qquad\square$

Remark 9.4 As we discussed above, Lemma 9.3 says that under the appropriate conditions, $\mu_{\lambda,w_{(3)}}^{(2)}$ and $\mu_{\lambda,w_{(1)}}^{(1)}$ satisfy natural analytic analogues of the $P(z_0)$-compatibility condition and the $P(z_2)$-compatibility condition, respectively. Note that from the proof of (9.2), $(Y'_{P(z_0)}(v,x_2^{-1})\mu_{\lambda,w_{(3)}}^{(2)})(w_{(1)}\otimes w_{(2)})$ "behaves qualitatively" like $\delta\left(\frac{z_2}{-x_2}\right)$ and so (9.10) "behaves qualitatively" like

$$x_1^{-1}\delta\left(\frac{x_2-z_0}{x_1}\right)\delta\left(\frac{z_2}{-x_2}\right),$$

suggesting the expected convergence when $|z_2| > |z_0|$. Analogously, from the proof of (9.4), $Y'_{P(z_2)}(v,x_0)\mu_{\lambda,w_{(1)}}^{(1)}(w_{(2)}\otimes w_{(3)})$ "behaves qualitatively" like $\delta\left(\frac{z_1}{x_0^{-1}}\right)$ and so (9.13) "behaves qualitatively" like

$$x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)\delta\left(\frac{z_1}{x_0^{-1}}\right),$$

again suggesting the expected convergence, this time when $|z_1| > |z_2|$. (By the $P(z_1,z_2)$-lower truncation condition, Res_{x_1} of (9.7) is upper-truncated in x_2, independently of $w_{(1)}$, $w_{(2)}$ and

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$w_{(3)}$, and Res_{x_2} of the first term on the right-hand side of (9.12), $(Y'_{P(z_1,z_2)}(v,x_0)\lambda)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$, is lower truncated in x_0, independently of the $w_{(j)}$.)

Remark 9.5 Given
$$\lambda \in (W_1 \otimes W_2 \otimes W_3)^*,$$

$w_{(1)} \in W_1$ and $w_{(3)} \in W_3$, it is also natural to ask whether, under suitable conditions, the evaluations $\mu^{(1)}_{\lambda,w_{(1)}} \in (W_2 \otimes W_3)^*$ and $\mu^{(2)}_{\lambda,w_{(3)}} \in (W_1 \otimes W_2)^*$ of λ satisfy the $P(z_2)$-local grading restriction condition and the $P(z_0)$-local grading restriction condition, respectively. In general, even for λ obtained from a product or an iterate of intertwining maps, these conditions are not satisfied by $\mu^{(1)}_{\lambda,w_{(1)}}$ or $\mu^{(2)}_{\lambda,w_{(3)}}$, but as we will see below, for such λ, the evaluations $\mu^{(1)}_{\lambda,w_{(1)}}$ and $\mu^{(2)}_{\lambda,w_{(3)}}$ satisfy certain analytic analogues of these conditions. These analogues motivate the next four important conditions on $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$. On the spaces (5.142) and (5.143) for $W_1 \otimes W_2$ and for $W_2 \otimes W_3$, the considerations of Remark 2.21 concerning the semisimple part of $L(0)_s$ of $L(0)$ hold, and in particular, on these spaces,

$$[L'_{P(z)}(0) - L'_{P(z)}(0)_s, L'_{P(z)}(0)] = 0$$

and so

$$[L'_{P(z)}(0) - L'_{P(z)}(0)_s, L'_{P(z)}(0)_s] = 0.$$

Hence

$$e^{yL'_{P(z)}(0)} = e^{yL'_{P(z)}(0)_s}e^{y(L'_{P(z)}(0)-L'_{P(z)}(0)_s)},$$

where y is a formal variable, and

$$e^{z'L'_{P(z)}(0)} = e^{z'L'_{P(z)}(0)_s}e^{z'(L'_{P(z)}(0)-L'_{P(z)}(0)_s)}$$

for $z' \in \mathbb{C}$. (A complex number denoted z' or $-z'$ will play this role in the considerations below.) Thus $e^{z'L'_{P(z)}(0)}$ maps any $P(z)$-generalized weight vector ν in $(W_2 \otimes W_3)^*$ or $(W_1 \otimes W_2)^*$ to a $P(z)$-generalized weight vector of the same generalized weight. An element $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ satisfying one of the conditions below means essentially that either $\mu^{(1)}_{\lambda,w_{(1)}} \in (W_2 \otimes W_3)^*$ or $\mu^{(2)}_{\lambda,w_{(3)}} \in (W_1 \otimes W_2)^*$ is the value at $z' = 0$ of the sum of a series, weakly absolutely convergent in the sense of Remark 7.24 for z' in a neighborhood of $z' = 0$, rather than just a finite sum, of the images under the map $e^{z'L'_{P(z)}(0)}$ of $P(z)$-generalized weight vectors or of ordinary weight vectors satisfying the $P(z)$-local grading restriction condition or the $L(0)$-semisimple $P(z)$-local grading restriction condition in Section 5, and that all of the summands lie in the same subspace whose grading is restricted. (For the four stronger conditions, the restriction on the grading is analogous to (2.89).) We shall typically use these conditions for $z = z_2$ when we consider $\mu^{(1)}_{\lambda,w_{(1)}}$ and for $z = z_0$ when we consider $\mu^{(2)}_{\lambda,w_{(3)}}$.

Recall the spaces (5.142) and (5.143) and recall that

$$e^{yL'_{P(z)}(0)} = e^{yL'_{P(z)}(0)_s}e^{y(L'_{P(z)}(0)-L'_{P(z)}(0)_s)}$$

and

$$e^{z' L'_{P(z)}(0)} = e^{z' L'_{P(z)}(0)_s} e^{z'(L'_{P(z)}(0) - L'_{P(z)}(0)_s)}$$

on these spaces; on the spaces (5.143),

$$L'_{P(z)}(0)_s = L'_{P(z)}(0).$$

Consider formal series $\sum_{n \in \mathbb{C}} \lambda_n^{(1)}$ and $\sum_{n \in \mathbb{C}} \lambda_n^{(2)}$ with

$$\lambda_n^{(1)} \in \coprod_{\beta \in \tilde{A}} ((W_2 \otimes W_3)^*)_{[n]}^{(\beta)}$$

and

$$\lambda_n^{(2)} \in \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)}$$

for $n \in \mathbb{C}$. Then there exist $K_n^{(1)}, K_n^{(2)} \in \mathbb{N}$ for $n \in \mathbb{C}$ such that

$$
\begin{aligned}
\sum_{n \in \mathbb{C}} e^{y L'_{P(z)}} \lambda_n^{(1)} &= \sum_{n \in \mathbb{C}} e^{y(L'_{P(z)}(0) - L'_{P(z)}(0)_s)} e^{y L'_{P(z)}(0)_s} \lambda_n^{(1)} \\
&= \sum_{n \in \mathbb{C}} e^{ny} \left(\sum_{i=0}^{K_n^{(1)}} \frac{y^i}{i!} (L'_{P(z)}(0) - n)^i \lambda_n^{(1)} \right)
\end{aligned}
\tag{9.14}
$$

and

$$
\begin{aligned}
\sum_{n \in \mathbb{C}} e^{y L'_{P(z)}} \lambda_n^{(2)} &= \sum_{n \in \mathbb{C}} e^{y(L'_{P(z)}(0) - L'_{P(z)}(0)_s)} e^{y L'_{P(z)}(0)_s} \lambda_n^{(2)} \\
&= \sum_{n \in \mathbb{C}} e^{ny} \left(\sum_{i=0}^{K_n^{(2)}} \frac{y^i}{i!} (L'_{P(z)}(0) - n)^i \lambda_n^{(2)} \right).
\end{aligned}
\tag{9.15}
$$

Remark 9.6 The formulas (9.14) and (9.15) also hold with y replaced by $z' \in \mathbb{C}$.

We shall be restricting our attention to $n \in \mathbb{R}$; we shall use the subspace

$$((W_1 \otimes W_2)^*)_{[\mathbb{R}]}^{(\tilde{A})} = \coprod_{n \in \mathbb{R}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)} \tag{9.16}$$

of $((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ and the correspondingly defined subspace

$$((W_1 \otimes W_2)^*)_{(\mathbb{R})}^{(\tilde{A})} = \coprod_{n \in \mathbb{R}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{(n)}^{(\beta)} \tag{9.17}$$

of $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$ (recall (5.142) and (5.143)), and similarly for $W_2 \otimes W_3$.

We next introduce the four conditions on

$$\lambda \in (W_1 \otimes W_2 \otimes W_3)^*.$$

These conditions are motivated by certain properties of such elements obtained from products or iterates of intertwining maps (see Proposition 9.13 below). Essentially there are really only two conditions, with the designations $P^{(1)}(z)$ and $P^{(2)}(z)$, but each of them has a non-semisimple version and a semisimple version. These four conditions will enter into the formulation of the "expansion condition."

The $P^{(1)}(z)$-local grading restriction condition

(a) The $P^{(1)}(z)$-*grading condition*: For any $w_{(1)} \in W_1$, there exists a formal series $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ with

$$\lambda_n^{(1)} \in \coprod_{\beta \in \tilde{A}} ((W_2 \otimes W_3)^*)_{[n]}^{(\beta)}$$

for $n \in \mathbb{R}$, an open neighborhood of $z' = 0$, and $N \in \mathbb{N}$ such that for $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, the series

$$\sum_{n \in \mathbb{R}} (e^{z' L'_{P(z)}(0)} \lambda_n^{(1)})(w_{(2)} \otimes w_{(3)})$$

(recall (9.14) and Remark 9.6; here we evaluate at $w_{(2)} \otimes w_{(3)}$) has the following properties:

(i) The series can be written as the iterated series

$$\sum_{n \in \mathbb{R}} e^{nz'} \left(\left(\sum_{i=0}^{N} \frac{(z')^i}{i!} (L'_{P(z)}(0) - n)^i \lambda_n^{(1)} \right) (w_{(2)} \otimes w_{(3)}) \right)$$

(recall from Proposition 7.8 that $\mathbb{R} \times \{0, \dots, N\}$ is a unique expansion set).

(ii) It is absolutely convergent for $z' \in \mathbb{C}$ in the open neighborhood of $z' = 0$ above.

(iii) It is absolutely convergent to $\mu_{\lambda, w_{(1)}}^{(1)}(w_{(2)} \otimes w_{(3)})$ when $z' = 0$:

$$\sum_{n \in \mathbb{R}} \lambda_n^{(1)}(w_{(2)} \otimes w_{(3)}) = \mu_{\lambda, w_{(1)}}^{(1)}(w_{(2)} \otimes w_{(3)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}).$$

(b) For any $w_{(1)} \in W_1$, let $W_{\lambda, w_{(1)}}^{(1)}$ be the smallest doubly graded (or equivalently, $\tilde{A}$-graded; recall Remark 5.40) subspace of $((W_2 \otimes W_3)^*)_{[\mathbb{R}]}^{(\tilde{A})}$, or equivalently, of $((W_2 \otimes W_3)^*)_{[\mathbb{C}]}^{(\tilde{A})}$, containing all the terms $\lambda_n^{(1)}$ in the formal series in (a) and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. (In view of Remark 5.42, $W_{\lambda, w_{(1)}}^{(1)}$

indeed exists, just as in the case of the $P(z)$-local grading restriction condition.) Then $W^{(1)}_{\lambda, w_{(1)}}$ has the properties

$$\dim(W^{(1)}_{\lambda, w_{(1)}})^{(\beta)}_{[n]} < \infty,$$

$$(W^{(1)}_{\lambda, w_{(1)}})^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \text{ sufficiently negative}$$

for any $n \in \mathbb{R}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{R}$-grading by $L'_{P(z)}(0)$-(generalized) eigenvalues and the superscripts denote the $\tilde{A}$-grading.

The $L(0)$-semisimple $P^{(1)}(z)$-local grading restriction condition

(a) The $L(0)$-*semisimple $P^{(1)}(z)$-grading condition*: For any $w_{(1)} \in W_1$, there exists a formal series $\sum_{n \in \mathbb{R}} \lambda^{(1)}_n$ with

$$\lambda^{(1)}_n \in \coprod_{\beta \in \tilde{A}} ((W_2 \otimes W_3)^*)^{(\beta)}_{(n)}$$

for $n \in \mathbb{R}$ and an open neighborhood of $z' = 0$ such that for $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, the series

$$\sum_{n \in \mathbb{R}} (e^{z' L'_{P(z)}(0)} \lambda^{(1)}_n)(w_{(2)} \otimes w_{(3)})$$

has the following properties:

(i) It can be written as

$$\sum_{n \in \mathbb{R}} e^{nz'} \lambda^{(1)}_n (w_{(2)} \otimes w_{(3)})$$

(recall from Proposition 7.8 that $\mathbb{R} \times \{0\}$ is a unique expansion set).

(ii) It is absolutely convergent for $z' \in \mathbb{C}$ in the neighborhood of $z' = 0$ above.

(iii) It is absolutely convergent to $\mu^{(1)}_{\lambda, w_{(1)}}(w_{(2)} \otimes w_{(3)})$ when $z' = 0$:

$$\sum_{n \in \mathbb{R}} \lambda^{(1)}_n (w_{(2)} \otimes w_{(3)}) = \mu^{(1)}_{\lambda, w_{(1)}}(w_{(2)} \otimes w_{(3)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}).$$

(Note that such an element λ also satisfies the $P^{(1)}(z)$-grading condition above with the same elements $\lambda^{(1)}_n$.)

(b) For any $w_{(1)} \in W_1$, consider the space $W^{(1)}_{\lambda, w_{(1)}}$ as above, which in this case is in fact the smallest doubly graded (or equivalently, $\tilde{A}$-graded) subspace of $((W_2 \otimes W_3)^*)^{(\tilde{A})}_{(\mathbb{R})}$ (or of $((W_2 \otimes W_3)^*)^{(\tilde{A})}_{(\mathbb{C})}$) containing all the terms $\lambda^{(1)}_n$ in the formal series in (a) and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$,

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$m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. Then $W^{(1)}_{\lambda, w_{(1)}}$ has the properties

$$\dim(W^{(1)}_{\lambda, w_{(1)}})^{(\beta)}_{(n)} < \infty,$$

$$(W^{(1)}_{\lambda, w_{(1)}})^{(\beta)}_{(n+k)} = 0 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{sufficiently negative}$$

for any $n \in \mathbb{R}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{R}$-grading by $L'_{P(z)}(0)$-eigenvalues and the superscripts denote the $\tilde{A}$-grading.

The $P^{(2)}(z)$-local grading restriction condition

(a) The $P^{(2)}(z)$-*grading condition*: For any $w_{(3)} \in W_3$, there exists a formal series $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ with

$$\lambda_n^{(2)} \in \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)^{(\beta)}_{[n]}$$

for $n \in \mathbb{R}$, an open neighborhood of $z' = 0$, and $N \in \mathbb{N}$ such that for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, the series

$$\sum_{n \in \mathbb{R}} (e^{z' L'_{P(z)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

has the following properties:

(i) It can be written as the iterated series

$$\sum_{n \in \mathbb{R}} e^{nz'} \left(\left(\sum_{i=0}^{N} \frac{(z')^i}{i!} (L'_{P(z)}(0) - n)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right)$$

(recall that $\mathbb{R} \times \{0, \ldots, N\}$ is a unique expansion set).

(ii) It is absolutely convergent for $z' \in \mathbb{C}$ in the neighborhood of $z' = 0$ above.

(iii) It is absolutely convergent to $\mu^{(2)}_{\lambda, w_{(3)}}(w_{(1)} \otimes w_{(2)})$ when $z' = 0$:

$$\sum_{n \in \mathbb{R}} \lambda_n^{(2)}(w_{(1)} \otimes w_{(2)}) = \mu^{(2)}_{\lambda, w_{(3)}}(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}).$$

(b) For any $w_{(3)} \in W_3$, let $W^{(2)}_{\lambda, w_{(3)}}$ be the smallest doubly graded (or equivalently, $\tilde{A}$-graded) subspace of $((W_1 \otimes W_2)^*)^{(\tilde{A})}_{[\mathbb{R}]}$, or equivalently, of $((W_1 \otimes W_2)^*)^{(\tilde{A})}_{[\mathbb{C}]}$, containing all the terms $\lambda_n^{(2)}$ in the formal series in (a) and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. (As above, $W^{(2)}_{\lambda, w_{(3)}}$ indeed exists.) Then $W^{(2)}_{\lambda, w_{(3)}}$ has the properties

$$\dim(W^{(2)}_{\lambda, w_{(3)}})^{(\beta)}_{[n]} < \infty,$$

$$(W^{(2)}_{\lambda, w_{(3)}})^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{sufficiently negative}$$

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for any $n \in \mathbb{R}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{R}$-grading by $L'_{P(z)}(0)$-(generalized) eigenvalues and the superscripts denote the $\tilde{A}$-grading.

The $L(0)$-semisimple $P^{(2)}(z)$-local grading restriction condition

(a) The $L(0)$-*semisimple $P^{(2)}(z)$-grading condition*: For any $w_{(3)} \in W_3$, there exists a formal series $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ with

$$\lambda_n^{(2)} \in \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{(n)}^{(\beta)}$$

for $n \in \mathbb{R}$ and an open neighborhood of $z' = 0$ such that for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, the series

$$\sum_{n \in \mathbb{R}} (e^{z' L'_{P(z)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

has the following properties:

(i) It can be written as

$$\sum_{n \in \mathbb{R}} e^{nz'} \lambda_n^{(2)}(w_{(1)} \otimes w_{(2)})$$

(recall that $\mathbb{R} \times \{0\}$ is a unique expansion set).

(ii) It is absolutely convergent for $z' \in \mathbb{C}$ in the neighborhood of $z' = 0$ above.

(iii) It is absolutely convergent to $\mu_{\lambda, w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)})$ when $z' = 0$:

$$\sum_{n \in \mathbb{R}} \lambda_n^{(2)}(w_{(1)} \otimes w_{(2)}) = \mu_{\lambda, w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}).$$

(Note that such an element λ also satisfies the $P^{(2)}(z)$-grading condition above with the same elements $\lambda_n^{(2)}$.)

(b) For any $w_{(3)} \in W_3$, consider the space $W_{\lambda, w_{(3)}}^{(2)}$ as above, which in this case is in fact the smallest doubly graded (or equivalently, $\tilde{A}$-graded) subspace of $((W_1 \otimes W_2)^*)_{(\mathbb{R})}^{(\tilde{A})}$ (or of $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$) containing all the terms $\lambda_n^{(2)}$ in the formal series in (a) and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. Then $W_{\lambda, w_{(3)}}^{(2)}$ has the properties

$$\dim(W_{\lambda, w_{(3)}}^{(2)})_{(n)}^{(\beta)} < \infty,$$

$$(W_{\lambda, w_{(3)}}^{(2)})_{(n+k)}^{(\beta)} = 0 \quad \text{for} \quad k \in \mathbb{Z} \text{ sufficiently negative}$$

for any $n \in \mathbb{R}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{R}$-grading by $L'_{P(z)}(0)$-eigenvalues and the superscripts denote the $\tilde{A}$-grading.

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Remark 9.7 Part (a) of each of these conditions says in particular that there exists $N \in \mathbb{N}$ ($N = 0$ in the semisimple case) such that when applied to an arbitrary element of $W_2 \otimes W_3$ or $W_1 \otimes W_2$, $(L'_{P(z)}(0) - L'_{P(z)}(0)_s)^{N+1} \lambda_n^{(1)}$ or $(L'_{P(z)}(0) - L'_{P(z)}(0)_s)^{N+1} \lambda_n^{(2)}$ becomes 0 for $n \in \mathbb{R}$. Thus Part (a) of each of these conditions implies:

$$\sum_{n \in \mathbb{R}} (e^{y L'_{P(z)}(0)} \lambda_n^{(1)})(w_{(2)} \otimes w_{(3)})$$

$$= \sum_{n \in \mathbb{R}} e^{ny} \left(\left(\sum_{i=0}^{N} \frac{y^i}{i!} (L'_{P(z)}(0) - n)^i \lambda_n^{(1)} \right) (w_{(2)} \otimes w_{(3)}) \right)$$

or

$$\sum_{n \in \mathbb{R}} (e^{y L'_{P(z)}(0)} \lambda_n^{(1)})(w_{(2)} \otimes w_{(3)}) = \sum_{n \in \mathbb{R}} e^{ny} \lambda_n^{(1)}(w_{(2)} \otimes w_{(3)})$$

or

$$\sum_{n \in \mathbb{R}} (e^{y L'_{P(z)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} e^{ny} \left(\left(\sum_{i=0}^{N} \frac{y^i}{i!} (L'_{P(z)}(0) - n)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right)$$

or

$$\sum_{n \in \mathbb{R}} (e^{y L'_{P(z)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}) = \sum_{n \in \mathbb{R}} e^{ny} \lambda_n^{(2)}(w_{(1)} \otimes w_{(2)}).$$

Part (a) of each of these conditions also asserts in particular, in the language of weak absolute convergence (recall Remark 7.24), that, for example, for any $w_{(1)} \in W_1$, $\mu_{\lambda, w_{(1)}}^{(1)}$ is the sum of a weakly absolutely convergent series $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ with $\lambda_n^{(1)} \in \coprod_{\beta \in \tilde{A}} ((W_2 \otimes W_3)^*)_{[n]}^{(\beta)}$.

While the grading restriction conditions above asserting the existence of the elements $\lambda_n^{(1)}$ or $\lambda_n^{(2)}$ do not say anything about the uniqueness of these elements, they are indeed unique:

Proposition 9.8 *The elements $\lambda_n^{(1)}$, $n \in \mathbb{R}$, in the $P^{(1)}(z)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z)$-local grading restriction condition) and the elements $\lambda_n^{(2)}$, $n \in \mathbb{R}$, in the $P^{(2)}(z)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z)$-local grading restriction condition) are uniquely determined by the properties indicated in Part (a) of the conditions.*

Since the proof of this result follows easily from certain facts established in the proof of Theorem 9.17 below, we defer it to Remark 9.19. This uniqueness result implies the following bilinearity result for the elements $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$:

Corollary 9.9 *The set of $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ satisfying any of the four local grading restriction conditions forms a linear subspace. The elements $\lambda_n^{(1)}$, $n \in \mathbb{R}$, in the $P^{(1)}(z)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z)$-local grading restriction*

condition) are bilinear in λ and $w_{(1)}$, and the elements $\lambda_n^{(2)}$, $n \in \mathbb{R}$, in the $P^{(2)}(z)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z)$-local grading restriction condition) are bilinear in λ and $w_{(3)}$.

Proof We prove only the case of the $P^{(1)}(z)$-local grading restriction condition; the other cases are handled the same way.

We shall use the notation $\lambda_n^{(1)}(\lambda, w_{(1)})$, $n \in \mathbb{R}$, to denote $\lambda_n^{(1)}$ in the $P^{(1)}(z)$-local grading restriction condition to exhibit the dependence of these elements on λ and $w_{(1)}$. Let λ and $\tilde{\lambda}$ be elements of $(W_1 \otimes W_2 \otimes W_3)^*$ satisfying the $P^{(1)}(z)$-local grading restriction condition, $w_{(1)}$ and $\tilde{w}_{(1)}$ elements of W_1, and a, b, c and d complex numbers. Then the formal series

$$\sum_{n \in \mathbb{R}} (ac\lambda_n^{(1)}(\lambda, w_{(1)}) + ad\lambda_n^{(1)}(\lambda, \tilde{w}_{(1)}) + bc\lambda_n^{(1)}(\tilde{\lambda}, w_{(1)}) + bd\lambda_n^{(1)}(\tilde{\lambda}, \tilde{w}_{(1)}))$$

satisfies (i) (ii) and (iii) in Part (a) of the $P^{(1)}(z)$-local grading restriction condition for $a\lambda + b\tilde{\lambda} \in (W_1 \otimes W_2 \otimes W_3)^*$ and $cw_{(1)} + d\tilde{w}_{(1)} \in W_1$, where we use the intersection of the four open neighborhoods of $z' = 0$ and the maximum of the relevant nonnegative integers N. The summands (for $n \in \mathbb{R}$) also satisfy Part (b) of the condition. Thus $a\lambda + b\tilde{\lambda}$ satisfies the $P^{(1)}(z)$-local grading restriction condition, and by Proposition 9.8 we have

$$\lambda_n^{(1)}(a\lambda + b\tilde{\lambda}, cw_{(1)} + d\tilde{w}_{(1)})$$
$$= ac\lambda_n^{(1)}(\lambda, w_{(1)}) + ad\lambda_n^{(1)}(\lambda, \tilde{w}_{(1)}) + bc\lambda_n^{(1)}(\tilde{\lambda}, w_{(1)}) + bd\lambda_n^{(1)}(\tilde{\lambda}, \tilde{w}_{(1)})$$

for $n \in \mathbb{R}$. $\square$

Using the uniqueness and recalling the $\tilde{A}$-homogeneous subspaces (5.88) and (8.43), we obtain the following natural $\tilde{A}$-properties of the elements λ_n in each of the four conditions:

Proposition 9.10 *Suppose that*

$$\lambda \in ((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta)},$$

with $\beta \in \tilde{A}$, satisfies the $P^{(1)}(z)$-local grading restriction condition, and suppose that

$$w_{(1)} \in W_1^{(\beta_1)},$$

with $\beta_1 \in \tilde{A}$. Then for each $n \in \mathbb{R}$,

$$\lambda_n^{(1)} \in ((W_2 \otimes W_3)^*)_{[n]}^{(\beta+\beta_1)}.$$

The analogous statement holds for each of the other three conditions; for instance, if

$$\lambda \in ((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta)}$$

satisfies the $L(0)$-semisimple $P^{(2)}(z)$-local grading restriction condition and

$$w_{(3)} \in W_3^{(\beta_3)},$$

with $\beta_3 \in \tilde{A}$, then

$$\lambda_n^{(2)} \in ((W_1 \otimes W_2)^*)_{(n)}^{(\beta+\beta_3)}.$$

Proof We prove only the first case mentioned, the proofs in the other cases being similar.
We have

$$\mu^{(1)}_{\lambda, w_{(1)}} \in ((W_2 \otimes W_3)^*)^{(\beta + \beta_1)},$$

since for $\beta_2, \beta_3 \in \tilde{A}$ such that $\beta_2 + \beta_3 \neq -\beta - \beta_1$ and for $w_{(2)} \in W_2^{(\beta_2)}$, $w_{(3)} \in W_3^{(\beta_3)}$,

$$\mu^{(1)}_{\lambda, w_{(1)}}(w_{(2)} \otimes w_{(3)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0,$$

so that we have the absolutely convergent sum

$$\sum_{n \in \mathbb{R}} \lambda^{(1)}_n (w_{(2)} \otimes w_{(3)}) = \mu^{(1)}_{\lambda, w_{(1)}}(w_{(2)} \otimes w_{(3)}) = 0.$$

For each $n \in \mathbb{R}$, let $\tilde{\lambda}^{(1)}_n$ be the projection of $\lambda^{(1)}_n$ to $((W_2 \otimes W_3)^*)^{(\beta + \beta_1)}_{[n]}$:

$$\tilde{\lambda}^{(1)}_n (w_{(2)} \otimes w_{(3)}) = \begin{cases} \lambda^{(1)}_n (w_{(2)} \otimes w_{(3)}) & \text{if } \beta_2 + \beta_3 = -\beta - \beta_1 \\ 0 & \text{if } \beta_2 + \beta_3 \neq -\beta - \beta_1. \end{cases}$$

Then clearly the $\tilde{\lambda}^{(1)}_n$ for $n \in \mathbb{R}$ also satisfy Part (a) of the $P^{(1)}(z)$-local grading restriction condition, and so by the uniqueness (Proposition 9.8),

$$\lambda^{(1)}_n = \tilde{\lambda}^{(1)}_n,$$

so that

$$\lambda^{(1)}_n \in ((W_2 \otimes W_3)^*)^{(\beta + \beta_1)}_{[n]}$$

for $n \in \mathbb{R}$. $\quad \square$

In the rest of this section, we shall focus on the case that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that the generalized V-modules that we start with are objects of $\mathcal{C}$. Recall the categories $\mathcal{M}_{sg}$ and $\mathcal{GM}_{sg}$ from Notation 2.36, and recall Assumptions 4.1, 5.30 and 7.11 on the category $\mathcal{C}$.

Remark 9.11 Let W_1, W_2, W_3 and W_4 be generalized V-modules. Given an $\tilde{A}$-compatible map

$$F : W_1 \otimes W_2 \otimes W_3 \to \overline{W}_4$$

as in Remark 8.12, there is a canonical $\tilde{A}$-compatible map G from W_4' to $(W_1 \otimes W_2 \otimes W_3)^*$ corresponding to F under the indicated canonical isomorphism between the spaces of such maps. We shall denote the map G corresponding to F by F':

$$F' = G : W_4' \to (W_1 \otimes W_2 \otimes W_3)^*.$$

Assume the convergence condition for intertwining maps in $\mathcal{C}$ and that all generalized V-modules considered are objects of $\mathcal{C}$. Let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and

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$P(z_0)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Then the maps

$$(I_1 \circ (1_{W_1} \otimes I_2))' : W_4' \to (W_1 \otimes W_2 \otimes W_3)^* \tag{9.18}$$

for $|z_1| > |z_2| > 0$ and

$$(I^1 \circ (I^2 \otimes 1_{W_3}))' : W_4' \to (W_1 \otimes W_2 \otimes W_3)^* \tag{9.19}$$

for $|z_2| > |z_0| > 0$ are well-defined $\tilde{A}$-compatible maps. In particular, for $w_{(4)}' \in W_4'$, we have the elements

$$(I_1 \circ (1_{W_1} \otimes I_2))'(w_{(4)}') \in (W_1 \otimes W_2 \otimes W_3)^* \tag{9.20}$$

and

$$(I^1 \circ (I^2 \otimes 1_{W_3}))'(w_{(4)}') \in (W_1 \otimes W_2 \otimes W_3)^*. \tag{9.21}$$

Proposition 9.10 applies to such elements λ when $w_{(4)}'$ is $\tilde{A}$-homogeneous, since for

$$w_{(4)}' \in (W_4')^{(\beta_4)},$$

with $\beta_4 \in \tilde{A}$, we have

$$\lambda \in ((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta_4)}$$

for λ either of the elements (9.20), (9.21), by the $\tilde{A}$-compatibility. This yields the natural $\tilde{A}$-properties of the corresponding elements λ_n in each of the four conditions (the complex numbers z being chosen in the ways that they arise naturally in the theory):

Proposition 9.12 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds. Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$ and let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Let*

$$w_{(4)}' \in (W_4')^{(\beta_4)},$$

with

$$\beta_4 \in \tilde{A}.$$

Assume that $|z_1| > |z_2| > 0$ and let

$$\lambda = (I_1 \circ (1_{W_1} \otimes I_2))'(w_{(4)}').$$

Suppose that λ satisfies the $P^{(1)}(z_2)$-local grading restriction condition or, respectively, the $P^{(2)}(z_0)$-local grading restriction condition. Then for $w_{(1)} \in W_1^{(\beta_1)}$ or, respectively, $w_{(3)} \in W_3^{(\beta_3)}$, with $\beta_j \in \tilde{A}$, we have

$$\lambda_n^{(1)} \in ((W_2 \otimes W_3)^*)_{[n]}^{(\beta_4 + \beta_1)}$$

or, respectively,

$$\lambda_n^{(2)} \in ((W_1 \otimes W_2)^*)_{[n]}^{(\beta_4 + \beta_3)}$$

for each $n \in \mathbb{R}$. When $\mathcal{C}$ is in $\mathcal{M}_{sg}$, suppose instead that λ satisfies the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition or, respectively, the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition. Then for $w_{(1)}$ and $w_{(3)}$ as above, we have

$$\lambda_n^{(1)} \in ((W_2 \otimes W_3)^*)_{(n)}^{(\beta_4 + \beta_1)}$$

or, respectively,

$$\lambda_n^{(2)} \in ((W_1 \otimes W_2)^*)_{(n)}^{(\beta_4 + \beta_3)}$$

for each $n \in \mathbb{R}$. The analogous conclusions hold if, instead, $|z_2| > |z_0| > 0$ and $(I^1 \circ (I^2 \otimes 1_{W_3}))'(w_{(4)}')$ satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$) or the $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). $\square$

In the next result, we prove that for the product of a $P(z_1)$-intertwining map I_1 and a $P(z_2)$-intertwining map I_2, each element (9.20) of the image of the map (9.18) satisfies the $P^{(1)}(z_2)$-local grading restriction condition and that for the iterate of a $P(z_2)$-intertwining map I^1 and a $P(z_0)$-intertwining map I^2, each element (9.21) of the image of the map (9.19) satisfies the $P^{(2)}(z_0)$-local grading restriction condition.

Proposition 9.13 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds. Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$ and let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Let $w_{(4)}' \in W_4'$. If $|z_1| > |z_2| > 0$, then*

$$(I_1 \circ (1_{W_1} \otimes I_2))'(w_{(4)}') \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$), and if $|z_2| > |z_0| > 0$, then

$$(I^1 \circ (I^2 \otimes 1_{W_3}))'(w_{(4)}') \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). Moreover, suppose that $\mathcal{C}$ is closed under images (recall Definition 5.35). Let $w_{(1)} \in W_1$ and $w_{(3)} \in W_3$. Take

$$\lambda_n^{(1)} \in (W_2 \otimes W_3)^*$$

and

$$\lambda_n^{(2)} \in (W_1 \otimes W_2)^*,$$

$n \in \mathbb{R}$, to be the elements constructed in the proof below. Then the corresponding spaces

$$W_{(I_1 \circ (1_{W_1} \otimes I_2))'(w_{(4)}'), w_{(1)}}^{(1)} \subset (W_2 \otimes W_3)^*$$

and

$$W^{(2)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(3)}} \subset (W_1 \otimes W_2)^*$$

(constructed using these elements $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$), equipped with the vertex operator maps given by $Y'_{P(z_2)}$ and $Y'_{P(z_0)}$, respectively, and the operators $L'_{P(z_2)}(j)$ and $L'_{P(z_0)}(j)$, respectively, for $j = -1, 0, 1$, are generalized V-submodules of objects of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$ and $(W_1 \otimes W_2)^*$, respectively. In particular, for any $n \in \mathbb{R}$, the doubly-graded subspaces

$$W_{\lambda_n^{(1)}} \subset W^{(1)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(1)}}$$

and

$$W_{\lambda_n^{(2)}} \subset W^{(2)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(3)}}$$

(recall the notation W_λ in the $P(z)$-local grading restriction condition in Section 5) are also generalized V-submodules of objects of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$ and $(W_1 \otimes W_2)^*$, respectively; $W_{\lambda_n^{(1)}}$ is the generalized V-submodule generated by $\lambda_n^{(1)}$ (the smallest generalized V-submodule containing $\lambda_n^{(1)}$), and analogously for $\lambda_n^{(2)}$.

Proof Let $w_{(1)} \in W_1$. For $n \in \mathbb{R}$, let $m'_{(1),n} \in M_1^*$ be defined by

$$m'_{(1),n}(m_{(1)}) = \langle w'_{(4)}, I_1(w_{(1)} \otimes \pi_n(m_{(1)})) \rangle$$

for $m_{(1)} \in M_1$. Since $w_{(1)}$ and $w'_{(4)}$ are finite sums of $\tilde{A}$-homogeneous elements and I_1 is $\tilde{A}$-compatible, $m'_{(1),n}$ is also a finite sum of $\tilde{A}$-homogeneous elements. Since by definition, for $m_{(1)} \in (M_1)_{[m]}$, $m'_{(1),n}(m_{(1)}) = 0$ when $m \neq n$, we see that $m'_{(1),n} \in (M'_1)_{[n]}$.

Let

$$\lambda_n^{(1)} = m'_{(1),n} \circ I_2 \in (W_2 \otimes W_3)^*.$$

From Notation 5.25, we have $\lambda_n^{(1)} = I'_2(m'_{(1),n})$, where $I'_2 : M'_1 \to (W_2 \otimes W_3)^*$ is as indicated in Notation 5.25. Since $m'_{(1),n} \in (M'_1)_{[n]}$, by Proposition 5.33(b),

$$\lambda_n^{(1)} = I'_2(m'_{(1),n}) \in ((W_2 \otimes W_3)^*)_{[n]}$$

for $n \in \mathbb{R}$. In addition, since I'_2 is $\tilde{A}$-compatible and $m'_{(1),n}$ is a finite sum of $\tilde{A}$-homogeneous elements, $\lambda_n^{(1)}$ is also a finite sum of $\tilde{A}$-homogeneous elements. By Proposition 7.12, the set $\{(n, i) \in \mathbb{C} \times \mathbb{N} \mid (L(0) - n)^i (M'_1)_{[n]} \neq 0\}$ is included in a (unique expansion) set of the form $\mathbb{R} \times \{0, \ldots, N\}$, and its subset $\{(n, i) \in \mathbb{C} \times \mathbb{N} \mid (L'_{P(z_2)}(0) - n)^i \lambda_n^{(1)} \neq 0\}$ is included in the same set (recall that I'_2 intertwines the various actions, including those of $L(0)$ and $L'_{P(z_2)}(0)$).

When $|z_1| > |z_2| > 0$, the product of I_1 and I_2 exists. For $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ we have

$$\mu^{(1)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(1)}}(w_{(2)} \otimes w_{(3)})$$

$$= \langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle$$

$$= \sum_{n \in \mathbb{R}} \langle w'_{(4)}, I_1(w_{(1)} \otimes \pi_n(I_2(w_{(2)} \otimes w_{(3)}))) \rangle$$

$$= \sum_{n \in \mathbb{R}} m'_{(1),n}(I_2(w_{(2)} \otimes w_{(3)}))$$

$$= \sum_{n \in \mathbb{R}} \lambda_n^{(1)}(w_{(2)} \otimes w_{(3)}),$$

an absolutely convergent series.

Let $\mathcal{Y}_1 = \mathcal{Y}_{I_1,0}$ and $\mathcal{Y}_2 = \mathcal{Y}_{I_2,0}$ (recall Proposition 4.8) so that

$$\begin{aligned} I_1(w_{(1)} \otimes w) &= \mathcal{Y}_1(w_{(1)}, z_1)w, \\ I_2(w_{(2)} \otimes w_{(3)}) &= \mathcal{Y}_2(w_{(2)}, z_2)w_{(3)} \end{aligned}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w \in M_1$ (recall the "substitution" notation from (4.12), where we choose $p = 0$). By Proposition 5.33(b), the map I'_2 preserves generalized weights. For $z' \in \mathbb{C}$, we also have, using (3.61) (recall (3.8) and Remark 3.34),

$$e^{z'L(0)} I_2(w_{(2)} \otimes w_{(3)})$$

$$= y^{L(0)} \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \Big|_{y^m = e^{mz'},\, \log y = z',\, x_2^m = e^{m \log z_2},\, \log x_2 = \log z_2}$$

$$= \mathcal{Y}_2(y^{L(0)} w_{(2)}, x_2 y) y^{L(0)} w_{(3)} \Big|_{y^m = e^{mz'},\, \log y = z',\, x_2^m = e^{m \log z_2},\, \log x_2 = \log z_2}$$

$$= \mathcal{Y}_2(e^{z'L(0)} w_{(2)}, x) e^{z'L(0)} w_{(3)} \Big|_{x^m = e^{m((\log z_2) + z')},\, \log x = (\log z_2) + z'}.$$

Thus

$$\sum_{n \in \mathbb{R}} \left(e^{z' L'_{P(z_2)}(0)} \lambda_n^{(1)} \right)(w_{(2)} \otimes w_{(3)})$$

$$= \sum_{n \in \mathbb{R}} \left(e^{z' L'_{P(z_2)}(0)} (I'_2(m'_{(1),n})) \right)(w_{(2)} \otimes w_{(3)})$$

$$= \sum_{n \in \mathbb{R}} \left(I'_2(e^{z'L(0)} m'_{(1),n}) \right)(w_{(2)} \otimes w_{(3)})$$

$$= \sum_{n \in \mathbb{R}} (e^{z'L(0)} m'_{(1),n})(I_2(w_{(2)} \otimes w_{(3)}))$$

$$= \sum_{n \in \mathbb{R}} m'_{(1),n}(e^{z'L(0)} I_2(w_{(2)} \otimes w_{(3)}))$$

$$= \sum_{n \in \mathbb{R}} \langle w'_{(4)}, I_1(w_{(1)} \otimes \pi_n(e^{z'L(0)} I_2(w_{(2)} \otimes w_{(3)}))) \rangle$$

$$= \sum_{n \in \mathbb{R}} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot$$

$$\pi_n(\mathcal{Y}_2(e^{z'L(0)} w_{(2)}, x) e^{z'L(0)} w_{(3)}) \rangle \Big|_{x_1^m = e^{m \log z_1},\, \log x_1 = \log z_1,\, x^m = e^{m((\log z_2) + z')},\, \log x = (\log z_2) + z'}$$

and when z' is in a small open neighborhood of 0 such that in particular $|z_1| > |e^{z'} z_2| > 0$, this series is absolutely convergent for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, and $w'_{(4)} \in W'_4$ (note that there exists $p \in \mathbb{Z}$ such that $(\log z_2) + z' = l_p(z_2 e^{z'})$; in fact, if z_2 is not a positive real number, then $(\log z_2) + z' = \log(z_2 e^{z'})$ for z' sufficiently near 0 and if z_2 is a positive real number, then either $(\log z_2) + z' = \log(z_2 e^{z'})$ or $(\log z_2) + z' = l_{-1}(z_2 e^{z'})$ (recall (4.10))). Hence $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ satisfies the $P^{(1)}(z_2)$-grading condition.

We know that the map I'_2 preserves generalized weights, and I'_2 is also $\tilde{A}$-compatible. Thus the image under I'_2 of the generalized V-submodule of the generalized V-module M'_1 generated by the elements $m'_{(1),n}$ for $n \in \mathbb{R}$ (that is, the smallest (strongly $\tilde{A}$-graded) generalized V-submodule containing these elements) satisfies the two grading restriction conditions (5.144) and (5.145). Since $W^{(1)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(1)}}$ is this image, Part (b) of the $P^{(1)}(z_2)$-local grading restriction condition holds.

When $\mathcal{C}$ is in $\mathcal{M}_{sg}$, the same arguments show that $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ satisfies the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition.

Moreover, $I'_2(M'_1)$ and $W^{(1)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(1)}}$ are generalized V-modules, and so is $W_{\lambda^{(1)}_n}$ for each $n \in \mathbb{R}$. If $\mathcal{C}$ is closed under images, then $I'_2(M'_1)$ is an object of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$, since $M'_1 \in \mathrm{ob}\,\mathcal{C}$, and so $W^{(1)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(1)}}$ and $W_{\lambda^{(1)}_n}$ for each $n \in \mathbb{R}$ are generalized submodules of objects of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$.

Now we handle the other case analogously. Let $w_{(3)} \in W_3$. For $n \in \mathbb{R}$, let $m'_{(2),n} \in M_2^*$ be defined by

$$m'_{(2),n}(m_{(2)}) = \langle w'_{(4)}, I^1(\pi_n(m_{(2)}) \otimes w_{(3)}) \rangle$$

for $m_{(2)} \in M_2$. Then $m'_{(2),n}$ is a finite sum of $\tilde{A}$-homogeneous elements and is an element of $(M'_2)_{[n]}$. Let

$$\lambda^{(2)}_n = m'_{(2),n} \circ I^2 \in (W_1 \otimes W_2)^*.$$

Then

$$\lambda^{(2)}_n = (I^2)'(m'_{(2),n}) \in ((W_1 \otimes W_2)^*)_{[n]}$$

for $n \in \mathbb{R}$ and is a finite sum of $\tilde{A}$-homogeneous elements, where $(I^2)' : M'_2 \to (W_1 \otimes W_2)^*$ is as indicated in Notation 5.25. By Proposition 7.12, the set $\{(n,i) \in \mathbb{C} \times \mathbb{N} \mid (L(0) - n)^i (M'_2)_{[n]} \neq 0\}$ is included in a (unique expansion) set of the form $\mathbb{R} \times \{0, \ldots, N\}$, and its subset $\{(n,i) \in \mathbb{C} \times \mathbb{N} \mid (L'_{P(z_0)}(0) - n)^i \lambda^{(2)}_n \neq 0\}$ is included in the same set.

When $|z_2| > |z_0| > 0$, the iterate of I^1 and I^2 exists. For $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$
\begin{aligned}
\mu^{(2)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(3)}} &(w_{(1)} \otimes w_{(2)}) \\
&= \langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}) \rangle \\
&= \sum_{n \in \mathbb{R}} \langle w'_{(4)}, I^1(\pi_n(I^2(w_{(1)} \otimes w_{(2)})) \otimes w_{(3)}) \rangle \\
&= \sum_{n \in \mathbb{R}} m'_{(2),n}(I^2(w_{(1)} \otimes w_{(2)}))
\end{aligned}
$$

$$= \sum_{n \in \mathbb{R}} \lambda_n^{(2)}(w_{(1)} \otimes w_{(2)}).$$

Let $\mathcal{Y}^1 = \mathcal{Y}_{I^1,0}$ and $\mathcal{Y}^2 = \mathcal{Y}_{I^2,0}$ so that

$$
\begin{aligned}
I^1(w \otimes w_{(3)}) &= \mathcal{Y}^1(w, z_2)w_{(3)}, \\
I^2(w_{(1)} \otimes w_{(2)}) &= \mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}
\end{aligned}
$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w \in M_2$. By Proposition 5.33(b), the map $(I^2)'$ preserves generalized weights. For $z' \in \mathbb{C}$, we also have

$$e^{z'L(0)}I^2(w_{(1)} \otimes w_{(2)})$$

$$= y^{L(0)}\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}\Big|_{y^m = e^{mz'}, \, \log y = z', \, x_0^m = e^{m \log z_0}, \, \log x_0 = \log z_0}$$

$$= \mathcal{Y}^2(y^{L(0)}w_{(1)}, x_0 y)y^{L(0)}w_{(2)}\Big|_{y^m = e^{mz'}, \, \log y = z', \, x_0^m = e^{m \log z_0}, \, \log x_0 = \log z_0}$$

$$= \mathcal{Y}^2(e^{z'L(0)}w_{(2)}, x)e^{z'L(0)}w_{(2)}\Big|_{x^m = e^{m((\log z_0)+z')}, \, \log x = (\log z_0)+z'}.$$

Hence

$$\sum_{n \in \mathbb{R}}(e^{z'L'_{P(z_0)}(0)}\lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}}(e^{z'L'_{P(z_0)}(0)}((I^2)'(m'_{(2),n})))(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}}((I^2)'(e^{z'L(0)}m'_{(2),n}))(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}}(e^{z'L(0)}m'_{(2),n})(I^2(w_{(1)} \otimes w_{(2)}))$$

$$= \sum_{n \in \mathbb{R}}m'_{(2),n}(e^{z'L(0)}I^2(w_{(1)} \otimes w_{(2)}))$$

$$= \sum_{n \in \mathbb{R}}\langle w'_{(4)}, I^1(\pi_n(e^{z'L(0)}I^2(w_{(1)} \otimes w_{(2)})) \otimes w_{(3)})\rangle$$

$$= \sum_{n \in \mathbb{R}}\langle w'_{(4)}, \mathcal{Y}^1(\pi_n(\mathcal{Y}^2(e^{z'L(0)}w_{(1)}, x)e^{z'L(0)}w_{(2)}), x_2) \cdot$$

$$\cdot w_{(3)}\rangle\Big|_{x_2^m = e^{m \log z_2}, \, \log x_2 = \log z_2, \, x^m = e^{m((\log z_0)+z')}, \, \log x = (\log z_0)+z'},$$

which is absolutely convergent for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, and $w'_{(4)} \in W'_4$, when z' is in a small open neighborhood of 0 such that in particular, $|z_2| > |e^{z'}z_0| > 0$. Thus $(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)})$ satisfies the $P^{(1)}(z_0)$-grading condition.

Since the map $(I^2)'$ preserves generalized weights and is also $\tilde{A}$-compatible, the image under $(I^2)'$ of the (strongly $\tilde{A}$-graded) generalized V-submodule of the generalized V-module

M_2' generated by the elements $m_{(2),n}'$ for $n \in \mathbb{R}$ satisfies the two grading restriction conditions (5.144) and (5.145). Since $W^{(2)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w_{(4)}'),w_{(3)}}$ is this image, Part (b) of the $P^{(2)}(z_0)$-local grading restriction condition holds.

The rest of the proof proceeds as above. $\quad\square$

We will often need to prove that certain generalized V-modules (or ordinary V-modules), in the *original* sense of Definitions 2.9, 2.11 and 2.12 (*without* the assumption of being strongly graded) are indeed strongly graded, and the main nontrivial properties to verify will often be the grading-restriction conditions (2.85) and (2.86). Thus we shall find the following definition useful, in particular in the next result:

Definition 9.14 In the setting of Definition 2.25 (the definition of "strongly graded"), a generalized V-module or a V-module (not necessarily strongly graded, of course) is *doubly graded* if it satisfies all the conditions in Definition 2.25 except perhaps for (2.85) and (2.86). The doubly-graded generalized V-submodule *generated by* given elements of a doubly-graded generalized V-module is (of course) the smallest doubly-graded (or equivalently, $\tilde{A}$-graded) generalized V-submodule containing the elements; similarly for doubly-graded V-modules.

Remark 9.15 A doubly-graded generalized V-submodule of a generalized V-module is of course strongly graded; similarly for V-modules. (Recall that a generalized V-module and a V-module are $\tilde{A}$-graded, and in addition strongly graded, by our conventions.)

Remark 9.16 Such structures have arisen in Propositions 5.33 and 5.67.

In general, for the product of a $P(z_1)$-intertwining map I_1 and a $P(z_2)$-intertwining map I_2, the elements of the image of the map (9.18) might not satisfy the $P^{(2)}(z_0)$-local grading restriction condition and for the iterate of a $P(z_2)$-intertwining map I^1 and a $P(z_0)$-intertwining map I^2, the elements of the image of the map (9.19) might not satisfy the $P^{(1)}(z_2)$-local grading restriction condition. But if they do, we have important consequences. In the theorem below, we shall prove a fundamental consequence, which plays an essential role in the rest of this section and in our construction of the associativity isomorphisms in the next section. The content of this theorem is essentially this: Given a product, and assuming the relevant condition, we construct a certain generalized V-module which will become (by virtue of Lemma 9.22 below) an intermediate module for a suitable iterate. This will allow us to express the product as an iterate, and vice versa when we start with an iterate. The hard part of the proof is to show that each term in the series given by the $P^{(2)}(z_0)$- or $P^{(1)}(z_2)$-local grading restriction condition satisfies the $P(z_0)$- or $P(z_2)$-compatibility condition, respectively.

Theorem 9.17 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that*
$$|z_1| > |z_2| > |z_0| > 0.$$

(Recall that $z_0 = z_1 - z_2$.) Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$ and let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Let $w'_{(4)} \in W'_4$.

1. *Suppose that $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ satisfies Part (a) of the $P^{(2)}(z_0)$-local grading restriction condition, that is, the $P^{(2)}(z_0)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For any $w_{(3)} \in W_3$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ be a series weakly absolutely convergent to*

$$\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}} \in (W_1 \otimes W_2)^*$$

   as indicated in the $P^{(2)}(z_0)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition), and suppose in addition that the elements $\lambda_n^{(2)} \in (W_1 \otimes W_2)^*$, $n \in \mathbb{R}$, satisfy the $P(z_0)$-lower truncation condition (Part (a) of the $P(z_0)$-compatibility condition in Section 5). Then each $\lambda_n^{(2)}$ satisfies the (full) $P(z_0)$-compatibility condition. Moreover, the corresponding space

$$W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}} \subset (W_1 \otimes W_2)^*,$$

   equipped with the vertex operator map given by $Y'_{P(z_0)}$ and the operators $L'_{P(z_0)}(j)$ for $j = -1, 0, 1$, is a doubly-graded generalized V-module, and when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, a doubly-graded V-module. In particular, if $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ satisfies the full $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$), then $W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}$ is a generalized V-module, that is, an object of $\mathcal{GM}_{sg}$ (or a V-module, that is, an object of $\mathcal{M}_{sg}$, when $\mathcal{C}$ is in $\mathcal{M}_{sg}$); in this case, the assumption that each $\lambda_n^{(2)}$ satisfies the $P(z_0)$-lower truncation condition is redundant.

2. *Analogously, suppose that $(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)})$ satisfies Part (a) of the $P^{(1)}(z_2)$-local grading restriction condition, that is, the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For any $w_{(1)} \in W_1$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ be a series weakly absolutely convergent to*

$$\mu^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}} \in (W_2 \otimes W_3)^*$$

   as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition), and suppose in addition that the elements $\lambda_n^{(1)} \in (W_2 \otimes W_3)^*$, $n \in \mathbb{R}$, satisfy the $P(z_2)$-lower truncation condition (Part (a) of the $P(z_2)$-compatibility condition). Then each $\lambda_n^{(1)}$ satisfies the (full) $P(z_2)$-compatibility condition. Moreover, the corresponding space

$$W^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}} \subset (W_2 \otimes W_3)^*,$$

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equipped with the vertex operator map given by $Y'_{P(z_2)}$ and the operators $L'_{P(z_2)}(j)$ for $j = -1, 0, 1$, is a doubly-graded generalized V-module, and when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, a doubly-graded V-module. In particular, if $(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)})$ satisfies the full $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$), then $W^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}}$ is a generalized V-module, that is, an object of $\mathcal{GM}_{sg}$ (or a V-module, that is, an object of $\mathcal{M}_{sg}$, when $\mathcal{C}$ is in $\mathcal{M}_{sg}$); in this case, the assumption that each $\lambda_n^{(1)}$ satisfies the $P(z_2)$-lower truncation condition is redundant.

Proof We will prove only Part 1 of the theorem, involving $I_1 \circ (1_{W_1} \otimes I_2)$; Part 2 is proved entirely analogously.

To prove that $W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}$ is a doubly-graded generalized V-module (and when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, a doubly-graded V-module), we claim that it is sufficient to prove that each $\lambda_n^{(2)}$, $n \in \mathbb{R}$, satisfies Part (b) of the $P(z_0)$-compatibility condition (and hence the $P(z_0)$-compatibility condition itself, since these elements are assumed to satisfy Part (a)). Indeed by Lemma 5.41, the space

$$(\mathrm{COMP}_{P(z_0)}((W_1 \otimes W_2)^*)) \cap ((W_1 \otimes W_2)^*)^{(\tilde{A})}$$

is $\tilde{A}$-graded, and hence so is its intersection

$$M = (\mathrm{COMP}_{P(z_0)}((W_1 \otimes W_2)^*)) \cap ((W_1 \otimes W_2)^*)^{(\tilde{A})}_{[\mathbb{C}]}$$

with the $\tilde{A}$-graded space $((W_1 \otimes W_2)^*)^{(\tilde{A})}_{[\mathbb{C}]}$. By Theorem 5.45, this space M is also $L'_{P(z_0)}(0)$-stable and hence $\mathbb{C}$-graded and therefore doubly graded. By Theorem 5.48 and Remark 5.42, M is a weak V-module and hence in fact a doubly-graded generalized V-module; when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, we replace the subscript $[\mathbb{C}]$ by $(\mathbb{C})$, and M is a doubly-graded V-module. By our hypothesis that each $\lambda_n^{(2)}$ satisfies the $P(z_0)$-compatibility condition, we have that $\lambda_n^{(2)} \in M$, and so

$$W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}} \subset M,$$

proving our claim.

The proof below of Part (b) of the $P(z_0)$-compatibility condition is a generalization of the proof of (14.51) in [H]. The proof here is (necessarily) much more elaborate. When $\mathcal{C}$ is in $\mathcal{M}_{sg}$, the proof below of course simplifies to a certain extent, but even in this case, our setting is more general than that in [H], and the proof here is correspondingly more delicate.

Let $\mathcal{Y}_1 = \mathcal{Y}_{I_1, 0}$ and $\mathcal{Y}_2 = \mathcal{Y}_{I_2, 0}$ (recall Proposition 4.8) so that

$$I_1(w_{(1)} \otimes w) = \mathcal{Y}_1(w_{(1)}, z_1)w, \tag{9.22}$$
$$I_2(w_{(2)} \otimes w_{(3)}) = \mathcal{Y}_2(w_{(2)}, z_2)w_{(3)} \tag{9.23}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w \in M_1$ (recall the "substitution" notation from (4.12), where we choose $p = 0$). For $z \in \mathbb{C}^\times$, let I_1^z and I_2^z be the $P(z_0 + zz_2)$- and

$P(zz_2)$-intertwining maps $I_{\mathcal{Y}_1,0}$ and $I_{\mathcal{Y}_2,0}$, respectively (assuming that $z_0 + zz_2 \neq 0$), so that

$$I_1^z(w_{(1)} \otimes w) = \mathcal{Y}_1(w_{(1)}, z_0 + zz_2)w, \qquad (9.24)$$

$$I_2^z(w_{(2)} \otimes w_{(3)}) = \mathcal{Y}_2(w_{(2)}, zz_2)w_{(3)} \qquad (9.25)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w \in M_1$; these maps are "deformations" of (9.22) and (9.23), which correspond to $z = 1$. Since $|z_1| > |z_2| > |z_0| > 0$, there exists a sufficiently small neighborhood of $z = 1$ such that

$$|z_0 + zz_2| > |zz_2| > |z_0| > 0$$

(recall that

$$z_1 = z_0 + z_2).$$

Since

$$\sum_{n \in \mathbb{R}} \langle w'_{(4)}, I_1^z(w_{(1)} \otimes \pi_n(I_2^z(w_{(2)} \otimes w_{(3)})))\rangle = \sum_{n \in \mathbb{R}} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_0 + zz_2)\pi_n(\mathcal{Y}_2(w_{(2)}, zz_2)w_{(3)})\rangle$$

$$(9.26)$$

is absolutely convergent when $|z_0 + zz_2| > |zz_2| > 0$ for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, the product $I_1^z \circ (1_{W_2} \otimes I_2^z)$ exists for z in a sufficiently small neighborhood of $z = 1$.

We shall establish a relationship between (9.26) and a certain Taylor series expansion in $\log z$.

The case $j = 0$, $z = z_0$ and $\lambda = \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}$ of (5.110) gives

$$(L'_{P(z_0)}(0)\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}})(w_{(1)} \otimes w_{(2)})$$

$$= \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}(w_{(1)} \otimes L(0)w_{(2)} + (L(0) + z_0 L(-1))w_{(1)} \otimes w_{(2)}). \quad (9.27)$$

Let x be a formal variable. Then recalling the notation from (4.12) with $p = 0$ and Remark 9.11, and using Definition 7.1, (9.22) and (9.23), we have

$$((1 - x)^{-L'_{P(z_0)}(0)}\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}})(w_{(1)} \otimes w_{(2)})$$

$$= (e^{-\log(1-x)L'_{P(z_0)}(0)}\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}})(w_{(1)} \otimes w_{(2)})$$

$$= \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}(e^{\log(1-x)(-z_0 L(-1) - L(0))}w_{(1)} \otimes e^{-\log(1-x)L(0)}w_{(2)})$$

$$= \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}((1 - x)^{-z_0 L(-1) - L(0)}w_{(1)} \otimes (1 - x)^{-L(0)}w_{(2)})$$

$$= ((I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}))((1 - x)^{-z_0 L(-1) - L(0)}w_{(1)} \otimes (1 - x)^{-L(0)}w_{(2)} \otimes w_{(3)})$$

$$= \langle w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))((1 - x)^{-z_0 L(-1) - L(0)}w_{(1)} \otimes (1 - x)^{-L(0)}w_{(2)} \otimes w_{(3)})\rangle$$

$$= \langle w'_{(4)}, I_1((1 - x)^{-z_0 L(-1) - L(0)}w_{(1)} \otimes I_2((1 - x)^{-L(0)}w_{(2)} \otimes w_{(3)}))\rangle$$

$$= \langle w'_{(4)}, \mathcal{Y}_1((1 - x)^{-z_0 L(-1) - L(0)}w_{(1)}, x_1)\mathcal{Y}_2((1 - x)^{-L(0)}w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_1 = z_1, \, x_2 = z_2},$$

$$(9.28)$$

where, because of Proposition 7.20, the coefficient of each power of x on the right-hand side of (9.28) has any of the meanings discussed in Remark 7.21, and in particular, means an absolutely convergent multisum or an analytic function of z_1 and z_2. The equality (9.28) says that the coefficient of each power of x on the left-hand side of (9.28) is equal to such an absolutely convergent multisum or such an analytic function obtained from the coefficient of the same power of x on the right-hand side.

Using Remark 3.42, we have

$$
\mathcal{Y}_1((1-x)^{-(x_1-x_2)L(-1)-L(0)}w_{(1)}, x_1)\mathcal{Y}_2((1-x)^{-L(0)}w_{(2)}, x_2)
$$
$$
= (1-x)^{-(L(0)-x_2L(-1))}\mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)(1-x)^{L(0)-x_2L(-1)}.
$$

Then by Proposition 7.20, we see that the right-hand side of (9.28) is equal to

$$
\langle w'_{(4)}, (1-x)^{-(L(0)-x_2L(-1))}\mathcal{Y}_1(w_{(1)}, x_1)\cdot
$$
$$
\left.\cdot\mathcal{Y}_2(w_{(2)}, x_2)(1-x)^{L(0)-x_2L(-1)}w_{(3)}\rangle_{W_4}\right|_{x_1=z_1,\ x_2=z_2}. \tag{9.29}
$$

Lemma 9.3 in [HL2], which used only the bracket formula for $L(0)$ and $L(-1)$, gives the formula

$$
(1-x)^{L(0)-x_2L(-1)} = e^{x_2xL(-1)}(1-x)^{L(0)},
$$

and so by (3.60), (9.29) is equal to

$$
\langle w'_{(4)}, (1-x)^{-L(0)}e^{-x_2xL(-1)}\mathcal{Y}_1(w_{(1)}, x_1)\cdot
$$
$$
\left.\cdot\mathcal{Y}_2(w_{(2)}, x_2)e^{x_2xL(-1)}(1-x)^{L(0)}w_{(3)}\rangle_{W_4}\right|_{x_1=z_1,\ x_2=z_2}
$$
$$
= \langle (1-x)^{-L'(0)}w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1 - x_2x)\cdot
$$
$$
\left.\cdot\mathcal{Y}_2(w_{(2)}, x_2 - x_2x)(1-x)^{L(0)}w_{(3)}\rangle_{W_4}\right|_{x_1=z_1,\ x_2=z_2}. \tag{9.30}
$$

Thus the left-hand side of (9.28) is equal to the right-hand side of (9.30) (as formal power series in x).

Let

$$
l^0(z) = \begin{cases} \log z & 0 \le \arg z < \pi \\ \log z - 2\pi i & \pi \le \arg z < 2\pi \end{cases}, \tag{9.31}
$$

which is a single-valued branch of the logarithm of z in the complex plane with a cut along the negative real line. We use this region to choose a branch because we will need a single-valued branch such that $z = 1$ is in the interior of the region. By Proposition 7.14 and (3.71),

$$
g(\zeta_1, \zeta_2, z) = \left.\langle e^{-l^0(z)L'(0)}w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)e^{l^0(z)L(0)}w_{(3)}\rangle_{W_4}\right|_{x_1=\zeta_1,\ x_2=\zeta_2} \tag{9.32}
$$

is a single-valued function of ζ_1, ζ_2 and z defined on $|\zeta_1| > |\zeta_2| > 0$, $\arg z \ne \pi$ and analytic when $\arg \zeta_1, \zeta_2 \ne 0$. The restriction of $g(\zeta_1, \zeta_2, z)$ to the subregion

$$
|\zeta_1| > |\zeta_2| > 0,\ 0 \le \arg \zeta_1, \arg \zeta_2 < \pi,\ \arg z \ne \pi
$$

can be analytically extended (i) to a single-valued function $g_1(\zeta_1, \zeta_2, z)$ defined on the regions given by $|\zeta_1| > |\zeta_2| > 0$, $\arg \zeta_1 \neq \pi$, $\arg z \neq \pi$ and analytic when $\arg \zeta_2 \neq 0$; (ii) to a single-valued function $g_2(\zeta_1, \zeta_2, z)$ defined on the region given by $|\zeta_1| > |\zeta_2| > 0$, $\arg \zeta_2 \neq \pi$, $\arg z \neq \pi$ and analytic when $\arg \zeta_1 \neq 0$; and (iii) to a single-valued analytic function $g_3(\zeta_1, \zeta_2, z)$ defined on the region given by $|\zeta_1| > |\zeta_2| > 0$, $\arg \zeta_1, \arg \zeta_2 \neq \pi$, $\arg z \neq \pi$ (recall Proposition 7.14). For convenience, we shall use $h(\zeta_1, \zeta_2, z)$ to denote $g(\zeta_1, \zeta_2, z)$ when $\arg z_1, \arg z_2 \neq 0$, to denote $g_1(\zeta_1, \zeta_2, z)$ when $\arg z_1 = 0$, $\arg z_2 \neq 0$, to denote $g_2(\zeta_1, \zeta_2, z)$ when $\arg z_1 \neq 0$, $\arg z_2 = 0$ and to denote $g_3(\zeta_1, \zeta_2, z)$ when $\arg z_1 = \arg z_2 = 0$. Then in particular, $h(\zeta_1, \zeta_2, z)$ is analytic near $\zeta_1 = z_1$, $\zeta_2 = z_2$ and $z = 1$ and we see that there exists a sufficiently small open neighborhood of $z = 1$ such that in this neighborhood, as the composition of the single-valued analytic function $h(\zeta_1, \zeta_2, z)$ of ζ_1, ζ_2 and z with the analytic functions

$$\zeta_1 = z_0 + z z_2$$

and

$$\zeta_2 = z z_2$$

of z, $h(z_0 + z z_2, z z_2, z)$ is a single-valued analytic function of z. For z satisfying

$$|z_0 + z z_2| > |z z_2| > 0,$$

$$0 \leq \arg(z_0 + z z_2), \arg(z z_2) < \pi, \tag{9.33}$$

$$\arg z \neq \pi,$$

by definition, we have

$$
\begin{aligned}
h(z_0 &+ z z_2, z z_2, z) \\
&= g(z_0 + z z_2, z z_2, z) \\
&= \langle e^{-l^0(z)L'(0)} w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) e^{l^0(z)L(0)} w_{(3)} \rangle_{W_4} \Big|_{x_1 = z_0 + z z_2,\, x_2 = z z_2}; \quad (9.34)
\end{aligned}
$$

moreover, π can be increased to 2π in (9.33) if $\arg z_1$ and/or $\arg z_2$ is positive, according to the cases discussed above.

In the region $\arg z \neq \pi$, the analytic function

$$z' = l^0(z)$$

is single-valued and univalent and $z' = 0$ is in the image of the region. So the composition of the function $h(z_0 + z z_2, z z_2, z)$ with the inverse function

$$z = e^{z'}$$

of the function $z' = l^0(z)$ gives us a single-valued analytic function

$$f(z') = h(z_0 + e^{z'} z_2, e^{z'} z_2, e^{z'}) \tag{9.35}$$

of z' in a sufficiently small open neighborhood of $z' = 0$. In particular, we can expand $f(z')$ as a power series in z'. Since the power series expansion of any function analytic at $z' = 0$ is uniquely determined by its derivatives at $z' = 0$, we can find the power series expansion of $f(z')$ as follows: Since $h(z_0 + zz_2, zz_2, z)$ is analytic at $z = 1$, we first expand it as a power series in $z - 1$ in a sufficiently small open neighborhood of $z = 1$. Then the power series expansion of $f(z')$ in a sufficiently small open neighborhood U of $z' = 0$ is obtained by replacing each nonnegative integral power of $z - 1$ by the corresponding power of $\sum_{k \in \mathbb{Z}_+} \frac{(z')^k}{k!}$. Since the convergence of the power series expansion of $f(z')$ is independent of $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, we can choose U to be independent of these elements.

We now want to give this power series explicitly (first, in powers of $z - 1$) using the right-hand side of (9.34). To do this, we have to restrict z to be in a subset such that $h(z_0 + zz_2, zz_2, z)$ is equal to the right-hand side of (9.34).

Let

$$O = e^U.$$

Then O is an open subset, containing 1, of the domain of the analytic function $h(z_0 + zz_2, zz_2, z)$ of z, and is independent of $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$. We choose U to be small enough so that the power series expansion of $h(z_0 + zz_2, zz_2, z)$ near $z = 1$ is absolutely convergent for $z \in O$.

Let P be (i) the set of $z \in \mathbb{C}$ with

$$0 \le \arg(z_0 + zz_2), \arg(zz_2) < 2\pi \tag{9.36}$$

when $\arg z_1, \arg z_2 \ne 0$ (so that in this case, P simply equals $\mathbb{C}$); or (ii) the set of z with

$$0 \le \arg(z_0 + zz_2) < \pi, \ 0 \le \arg(zz_2) < 2\pi$$

when $\arg z_1 = 0$ but $\arg z_2 \ne 0$; or (iii) the set of z with

$$0 \le \arg(z_0 + zz_2) < 2\pi, \ 0 \le \arg(zz_2) < \pi$$

when $\arg z_1 \ne 0$ but $\arg z_2 = 0$; or (iv) the set of z with

$$0 \le \arg(z_0 + zz_2), \arg(zz_2) < \pi$$

when $\arg z_1 = \arg z_2 = 0$. Then by definition, for

$$z \in O \cap P,$$

we have

$$h(z_0 + zz_2, zz_2, z) = g(z_0 + zz_2, zz_2, z), \tag{9.37}$$

which is given by the right-hand side of (9.34). Note that

$$1 \in O \cap P.$$

For
$$z' = l^0(z) \in U$$
($z \in O$), the power series expansion of $f(z')$ can be obtained explicitly from the right-hand side of (9.34) as follows:

By Propositions 7.12 and 7.20 (and in particular, (7.46)), (3.71) and Remark 7.21, one of the equivalent meanings of (9.32) is an absolutely convergent series in the region $|\zeta_1| > |\zeta_2| > 0$ of the form

$$\sum_{s=0}^{S}\sum_{t=0}^{T}\sum_{p,q\in\mathbb{R}}\sum_{k=0}^{K}\sum_{l=0}^{L} b_{s,t,p,q,k,l} e^{p\log\zeta_1}(\log\zeta_1)^k e^{q\log\zeta_2}(\log\zeta_2)^l e^{c_s l^0(z)}(l^0(z))^t, \tag{9.38}$$

where $b_{s,t,p,q,k,l} \in \mathbb{C}$ and $c_s \in \mathbb{R}$. By Lemma 7.7, the derivatives of (9.38) are given by the absolutely convergent series

$$\sum_{s=0}^{S}\sum_{t=0}^{T}\sum_{p,q\in\mathbb{R}}\sum_{k=0}^{K}\sum_{l=0}^{L} b_{s,t,p,q,k,l} \frac{\partial^i}{\partial\zeta_1^i}\frac{\partial^j}{\partial\zeta_2^j}\frac{\partial^n}{\partial z^n}\left(e^{p\log\zeta_1}(\log\zeta_1)^k e^{q\log\zeta_2}(\log\zeta_2)^l e^{c_s l^0(z)}(l^0(z))^t\right)$$
$$\tag{9.39}$$

for $i, j, n \in \mathbb{N}$.

On the other hand, the coefficients of the expansion of $h(z_0 + zz_2, zz_2, z)$ as a power series in $z - 1$ are given by its derivatives at $z = 1$. By the chain rule, there exist $\alpha_{m,i,j,n} \in \mathbb{C}$ (depending on z_2) such that for any analytic function $F(\zeta_1, \zeta_2, z)$ of ζ_1, ζ_2, z near $\zeta_1 = z_1$, $\zeta_2 = z_2$ and $z = 1$ (in particular, for $F(\zeta_1, \zeta_2, z) = h(\zeta_1, \zeta_2, z)$),

$$\frac{\partial^m}{\partial z^m}F(z_0 + zz_2, zz_2, z)\Big|_{z=1} = \sum_{i+j+n=m,\, i,j,n\in\mathbb{N}} \alpha_{m,i,j,n}\frac{\partial^i}{\partial\zeta_1^i}\frac{\partial^j}{\partial\zeta_2^j}\frac{\partial^n}{\partial z^n}F(\zeta_1, \zeta_2, z)\Big|_{\zeta_1=z_1,\, \zeta_2=z_2,\, z=1}. \tag{9.40}$$

For $z \in O \cap P$, $h(z_0 + zz_2, zz_2, z)$ is equal to the right-hand side of (9.34). But one of the meanings of the right-hand side of (9.34) is the absolutely convergent series

$$\sum_{s=0}^{S}\sum_{t=0}^{T}\sum_{p,q\in\mathbb{R}}\sum_{k=0}^{K}\sum_{l=0}^{L} b_{s,t,p,q,k,l} e^{p\log(z_0+zz_2)}(\log(z_0+zz_2))^k e^{q\log(zz_2)}(\log(zz_2))^l e^{c_s l^0(z)}(l^0(z))^t$$
$$\tag{9.41}$$

(see (9.32) and (9.38)). Using (9.39), (9.40) and (9.41), we see that the m-th derivative of $h(z_0 + zz_2, zz_2, z)$ at $z = 1$ is equal to the absolutely convergent series

$$\sum_{s=0}^{S}\sum_{t=0}^{T}\sum_{p,q\in\mathbb{R}}\sum_{k=0}^{K}\sum_{l=0}^{L} b_{s,t,p,q,k,l} \sum_{i+j+n=m,\, i,j,n\in\mathbb{N}} \alpha_{m,i,j,n}.$$
$$\cdot\frac{\partial^i}{\partial\zeta_1^i}\frac{\partial^j}{\partial\zeta_2^j}\frac{\partial^n}{\partial z^n}\left(e^{p\log\zeta_1}(\log\zeta_1)^k e^{q\log\zeta_2}(\log\zeta_2)^l e^{c_s l^0(z)}(l^0(z))^t\right)\Big|_{\zeta_1=z_1,\, \zeta_2=z_2,\, z=1} \tag{9.42}$$

for $m \in \mathbb{N}$. (Note that when $\arg z_1$ or $\arg z_2$ is 0, 1 is not in the interior of $O \cap P$ and we have to calculate the derivatives above using only $z \in O \cap P$. But the result is the same.)

Thus we see that the coefficients of the expansion of $h(z_0 + zz_2, zz_2, z)$ as a power series in $z - 1$ are given by (9.42) divided by $m!$.

By (9.40) and (9.41), we see that the coefficients of this power series in $z - 1$ are also equal to the coefficients of the power series in $z - 1$ obtained from (9.41) by replacing

$$
\begin{aligned}
e^{p \log(z_0 + zz_2)}(\log(z_0 + zz_2))^k &= e^{p \log(z_1 + (z-1)z_2)}(\log(z_1 + (z-1)z_2))^k, \\
e^{q \log(zz_2)}(\log(zz_2))^l &= e^{q \log(z_2 + (z-1)z_2)}(\log(z_2 + (z-1)z_2))^l, \\
e^{c_s l^0(z)}(l^0(z))^t &= e^{c_s l^0(1+(z-1))}(l^0(1+(z-1)))^t
\end{aligned}
\tag{9.43}
$$

by their power series expansions near $z = 1$. We have shown that the power series obtained in this way has the indicated sums of absolutely convergent series as coefficients and is absolutely convergent to $h(z_0 + zz_2, zz_2, z)$ when $z \in O$ (since we chose U and O small enough, above). We have succeeded in giving this power series in $z - 1$ explicitly. Finally, as above, we replace each nonnegative integral power of $z - 1$ by the corresponding power of $\sum_{k \in \mathbb{Z}_+} \frac{(z')^k}{k!}$ to obtain the power series expansion of $f(z')$ for $z' \in U$.

Note that the constant terms of both the power series expansion of

$$
l^0(z) = l^0(1 + (z - 1))
$$

near $z = 1$ and the formal power series $\log(1 + x)$ are 0, and in fact, this expansion of $l^0(z)$ is obtained by substituting $z - 1$ for x in the formal series $\log(1 + x)$. (This is of course a reflection of the fact that the formal power series notation "$\log(1 + x)$" is in effect choosing a branch of a multivalued function.) Thus from the explicit expansion procedure obtaining the power series in $z - 1$ above and the precise meaning of the right-hand side of (9.30), we see that, as sums of absolutely convergent series, the coefficient of the n-th power of x in the formal power series in x given by the right-hand side of (9.30) is exactly the same as $(-1)^n$ times the coefficient of the n-th power of $z - 1$ in the power series in $z - 1$ obtained above. Thus, if we substitute $-(z - 1)$ for x in the right-hand side of (9.30), we obtain the explicit expansion above of the right-hand side of (9.34) as a power series in $z - 1$.

Because of the explicit calculations and discussions above, when $z' \in U$, the power series expansion of $f(z')$ can be obtained using the right-hand side of (9.30) and the following two steps: (i) Substitute $1 - e^y \in -y + y^2 \mathbb{C}[[y]]$ for x in the right-hand side of (9.30) and (ii) substitute z' for y in the resulting series. This power series in z' as the expansion of $f(z')$ must be absolutely convergent in the neighborhood U of $z' = 0$ and its sum is equal to the single-valued analytic function $h(z_0 + e^{z'}z_2, e^{z'}z_2, e^{z'})$. In particular, for $z' \in U$ and $z \in P$, this power series in $z' = l^0(z)$ is absolutely convergent to the right-hand side of (9.34).

Applying the same steps (i) and (ii) above to the left-hand side of (9.28), we also obtain a power series $S(z')$ in z'. Since the left-hand side of (9.28) is equal to the right-hand side of (9.30) as formal power series in x, we see that the power series expansion of $f(z')$ and the power series $S(z')$ are the same. In particular, in the neighborhood U of $z' = 0$, $S(z')$ is absolutely convergent to $f(z')$. Since $f(z')$ is equal to the right-hand side of (9.34) when $z' = l^0(z) \in U$ and $z \in P$, $S(z')$ is absolutely convergent to the right-hand side of (9.34) when $z' = l^0(z) \in U$ and $z \in P$, that is, when $z \in O \cap P$. (Recall from (9.36) that in case

$\arg z_1 \neq 0$ and $\arg z_2 \neq 0$, $P = \mathbb{C}$, so that in this case, $S(z')$ is absolutely convergent to the right-hand side of (9.34) whenever $z' \in U$.)

By assumption, for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, the series

$$\sum_{n \in \mathbb{R}} \lambda_n^{(2)}(w_{(1)} \otimes w_{(2)}) \tag{9.44}$$

converges absolutely to

$$\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}(w_{(1)} \otimes w_{(2)}),$$

and

$$\sum_{n \in \mathbb{R}} (e^{z' L'_{P(z_0)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

is absolutely convergent for z' in an open neighborhood of $z' = 0$ independent of $w_{(1)}$ and $w_{(2)}$. (Note that the elements $\lambda_n^{(2)}$ for $n \in \mathbb{R}$ depend on $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$.)

Let

$$Q = \{z \in \mathbb{C} \mid 0 \leq \arg z < \pi\}.$$

Note that for $z \in Q$,

$$l^0(z) = \log z.$$

We will show that the series

$$\sum_{n \in \mathbb{R}} (e^{-(l^0(z)) L'_{P(z_0)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}), \tag{9.45}$$

which is absolutely convergent in an open neighborhood of $z = 1$ independent of $w_{(1)}$ and $w_{(2)}$ and absolutely convergent to

$$\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}(w_{(1)} \otimes w_{(2)})$$

for $z = 1$, gives a double series of the form $\sum_{n \in \mathbb{R}} \sum_{i=0}^N a_{n,i} e^{-n \log z} (-\log z)^i$ absolutely convergent to $f(\log z)$ for z in a nonempty open subset of $O \cap P \cap Q$, and since by Proposition 7.8 $\mathbb{R} \times \{0, \ldots, N\}$ is a unique expansion set, the coefficients $a_{n,i}$ and related numbers are uniquely determined.

Since

$$\lambda_n^{(2)} \in \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)},$$

we have

$$L'_{P(z_0)}(0)_s \lambda_n^{(2)} = n \lambda_n^{(2)}$$

(recall Remarks 2.21 and 9.5).

In the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$, $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ satisfies the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition, so that

$$\lambda_n^{(2)} \in \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{(n)}^{(\beta)},$$

and we have

$$L'_{P(z_0)}(0)_s = L'_{P(z_0)}(0)$$

and

$$L'_{P(z_0)}(0)\lambda_n^{(2)} = n\lambda_n^{(2)}.$$

In particular, the proof below will give the desired result, and is in fact simpler, in this case.

From the $P^{(2)}(z_0)$-grading condition, we have, for z in an open neighborhood of 1 independent of $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, and with N independent of $w_{(1)}$ and $w_{(2)}$,

$$\sum_{n \in \mathbb{R}} (e^{-(l^0(z))L'_{P(z_0)}(0)}\lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} e^{-n(l^0(z))} \left(\left(\sum_{i=0}^{N} \frac{(-l^0(z))^i}{i!}(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right).$$

(9.46)

The derivative with respect to $z' = l^0(z)$ of the iterated series (9.46) is

$$\sum_{n \in \mathbb{R}} \frac{\partial}{\partial l^0(z)}(e^{-(l^0(z))L'_{P(z_0)}(0)}\lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} \frac{\partial}{\partial l^0(z)} \left(e^{-n(l^0(z))} \left(\left(\sum_{i=0}^{N} \frac{(-l^0(z))^i}{i!}(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right) \right)$$

$$= \sum_{n \in \mathbb{R}} (-n)e^{-n(l^0(z))} \left(\left(\sum_{i=0}^{N} \frac{(-l^0(z))^i}{i!}(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right)$$

$$+ \sum_{n \in \mathbb{R}} e^{-n(l^0(z))} \left(\left(\sum_{i=1}^{N} \frac{-(-l^0(z))^{i-1}}{(i-1)!}(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right)$$

$$= -\sum_{n \in \mathbb{R}} e^{-n(l^0(z))} \cdot$$

$$\cdot \left(\left(\sum_{i=0}^{N} \frac{(-l^0(z))^i}{i!}(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i L'_{P(z_0)}(0)_s \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right)$$

$$- \sum_{n \in \mathbb{R}} e^{-n(l^0(z))} \left(\left(\sum_{i=0}^{N} \frac{(-l^0(z))^i}{i!}(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^{i+1} \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right)$$

$$= -\sum_{n \in \mathbb{R}} e^{-n(l^0(z))} \cdot$$

$$\cdot \left(\left(\sum_{i=0}^{N} \frac{(-l^0(z))^i}{i!} L'_{P(z_0)}(0)(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right)$$

$$= -\sum_{n \in \mathbb{R}} (L'_{P(z_0)}(0)e^{-(l^0(z))L'_{P(z_0)}(0)}\lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

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$$= -\sum_{n\in\mathbb{R}}(e^{-(l^0(z))L'_{P(z_0)}(0)}\lambda_n^{(2)})(w_{(1)} \otimes L(0)w_{(2)} + (L(0) + z_0 L(-1))w_{(1)} \otimes w_{(2)}) \qquad (9.47)$$

(recall (5.110)). Since (9.45) is absolutely convergent for $z' = l^0(z)$ in an open neighborhood of $z' = 0$ independent of $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, so is the left-hand side of (9.46). Thus the right-hand side of (9.47) and consequently the left-hand side of (9.47) is absolutely convergent for $z' = l^0(z)$ in the same neighborhood of $z' = 0$. Since the map l^0 is univalent in a neighborhood of $z = 1$, we see that there exists an open neighborhood Π of $z = 1$ independent of $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ such that both sides of (9.46) and of (9.47) are absolutely convergent. For later use, we may and do choose Π to be a small open disk centered at 1. The same calculation and argument show that all the higher derivatives with respect to $z' = l^0(z)$ of the iterated series (9.46) are also absolutely convergent for $z \in \Pi$.

Since $l^0(z) = \log z$ for $z \in Q$, we see by Proposition 7.9 that

$$\sum_{n\in\mathbb{R}} e^{-n(\log z)}((L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

is absolutely convergent for $z \in \Pi \cap Q$, for each $i = 0, \dots, N$. Since $n \in \mathbb{R}$, we have

$$|e^{-n(\log z)}| = |e^{-n(l^0(z))}| = |e^{-n(l^0(\bar{z}))}|$$

for $z \in \mathbb{C}^\times$, and since $\overline{\Pi} = \Pi$, for $z \in \Pi$,

$$\sum_{n\in\mathbb{R}} e^{-n(l^0(z))}((L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}) \qquad (9.48)$$

is absolutely convergent for $i = 0, \dots, N$. Thus the double series

$$\sum_{n\in\mathbb{R}} \sum_{i=0}^{N} e^{-n(l^0(z))} \frac{(-l^0(z))^i}{i!}((L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}) \qquad (9.49)$$

is absolutely convergent for $z \in \Pi$. Since for any $z \in \mathbb{C}^\times$,

$$\sum_{n\in\mathbb{R}} e^{-n(l^0(z))}((L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

can be written as a series of the form of $\sum_{n\in\mathbb{R}} a_n e^{-n(\log z)}$, by Lemma 7.7 the sums of (9.48) give analytic functions of $z \in \Pi$ for $i = 0, \dots, N$. Thus the sum of (9.46), or equivalently, the sum of (9.49), gives an analytic function of $z \in \Pi$. Using (9.47) repeatedly, we see that for $k \in \mathbb{N}$, the k-th derivative with respective to $z' = l^0(z)$ of this analytic function is given by the absolutely convergent series

$$(-1)^k \sum_{n\in\mathbb{R}}((L'_{P(z_0)}(0))^k e^{-(l^0(z))L'_{P(z_0)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}) \qquad (9.50)$$

for $z \in \Pi$, and its k-th derivative with respect to z' at $z' = 0$ or equivalently at $z = 1$ is given by the absolutely convergent series

$$(-1)^k \sum_{n \in \mathbb{R}} ((L'_{P(z_0)}(0))^k \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}). \tag{9.51}$$

Since

$$\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$$

is weakly absolutely convergent to

$$\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}},$$

using (9.27) we have, starting as in (9.28),

$$
\begin{aligned}
&((1-x)^{-L'_{P(z_0)}(0)} \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}})(w_{(1)} \otimes w_{(2)}) \\
&= (e^{-\log(1-x) L'_{P(z_0)}(0)} \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}})(w_{(1)} \otimes w_{(2)}) \\
&= \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}(e^{\log(1-x)(-z_0 L(-1) - L(0))} w_{(1)} \otimes e^{-\log(1-x) L(0)} w_{(2)}) \\
&= \sum_{n \in \mathbb{R}} \lambda_n^{(2)} (e^{\log(1-x)(-z_0 L(-1) - L(0))} w_{(1)} \otimes e^{-\log(1-x) L(0)} w_{(2)}) \\
&= \sum_{n \in \mathbb{R}} (e^{-\log(1-x) L'_{P(z_0)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}) \\
&= \sum_{n \in \mathbb{R}} ((1-x)^{-L'_{P(z_0)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}), \tag{9.52}
\end{aligned}
$$

where the absolute convergence holds for the coefficient of each power of x in the formal power series in x in (9.52). We also have

$$
\begin{aligned}
&(1-x)^{-L'_{P(z_0)}(0)} \lambda_n^{(2)} \\
&= e^{-\log(1-x) L'_{P(z_0)}(0)} \lambda_n^{(2)} \\
&= e^{-\log(1-x)(L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)} e^{-\log(1-x) L'_{P(z_0)}(0)_s} \lambda_n^{(2)} \\
&= e^{-n \log(1-x)} \sum_{i=0}^{K_n} \frac{(-\log(1-x))^i}{i!} (L'_{P(z_0)}(0) - L'_{P(z_0)}(0)_s)^i \lambda_n^{(2)} \\
&= (1-x)^{-n} \sum_{i=0}^{K_n} \frac{(-\log(1-x))^i}{i!} (L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)} \tag{9.53}
\end{aligned}
$$

where $K_n \in \mathbb{N}$; cf. (9.15). From (9.52) and (9.53), we obtain

$$((1-x)^{-L'_{P(z_0)}(0)} \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}})(w_{(1)} \otimes w_{(2)})$$

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$$= \sum_{n \in \mathbb{R}} ((1-x)^{-L'_{P(z_0)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} (1-x)^{-n} \sum_{i=0}^{K_n} \frac{(-\log(1-x))^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}),$$

$$(9.54)$$

with absolute convergence for each power of x, as in (9.52).

Recall that we have proved that if we substitute $1 - e^y$ for x in the left-hand side of (9.28), which is the same as the left-hand side of (9.54), and then substitute $l^0(z)$ for y, we obtain an absolutely convergent power series $S(l^0(z))$ in $l^0(z)$ for $z \in O$, and for $z \in O \cap Q$, so that $l^0(z) = \log z$, the sum of this series $S(\log z)$ is equal to $f(\log z)$, that is, if we also use $S(\log z)$ to denote its sum, then

$$S(\log z) = f(\log z)$$

(recall (9.35)). Moreover, for

$$z \in O \cap P \cap Q,$$

this is also equal to the right-hand side of (9.34) (recall (9.37)).

The same substitution steps in the right-hand side of (9.54) give the same absolutely convergent series $S(l^0(z))$. Substituting $1 - e^y$ for x in (9.54) and using Remark 9.7, we obtain

$$(e^{-yL'_{P(z_0)}(0)} \mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} (e^{-yL'_{P(z_0)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} e^{-ny} \sum_{i=0}^{N} \frac{(-y)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}), \qquad (9.55)$$

where the absolute convergence holds for the coefficient of each power of y in the formal power series in y in (9.55). Thus $S(l^0(z))$ is equal to the series obtained by substituting $l^0(z)$ for y in (9.55). In particular, for $k \in \mathbb{N}$, the k-th derivative with respect to $l^0(z)$ of $S(l^0(z))$ at $l^0(z) = 0$ (or equivalently at $z = 1$) is equal to the constant term of the k-th derivative with respect to y of (9.55), and this is equal to (9.51). We know that $S(l^0(z))$ is an absolutely convergent power series in $l^0(z)$ for $z \in O$. In particular, for $k \in \mathbb{N}$, the sum of the k-th derivative at $l^0(z) = 0$ of the series $S(l^0(z))$ is equal to the k-th derivative at $l^0(z) = 0$ of the analytic function given by the sum of $S(l^0(z))$. Since for $k \in \mathbb{N}$, the k-th derivative at $l^0(z) = 0$ of the analytic function given by the sum of (9.46) and the k-th derivative at $l^0(z) = 0$ of the analytic function given by the sum of $S(l^0(z))$ are equal, these two analytic functions must be equal on an open neighborhood Γ of $z = 1$ in the intersection of their domains. Clearly we can choose Γ to be independent of $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Then for

$$z \in O \cap P \cap Q \cap \Pi \cap \Gamma,$$

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$$\sum_{n \in \mathbb{R}} e^{-n \log z} \left(\sum_{i=0}^{N} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}) \right) \tag{9.56}$$

is absolutely convergent to the right-hand side of (9.34) and in fact, we have proved that the corresponding double series, namely,

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}), \tag{9.57}$$

is also absolutely convergent to the right-hand side of (9.34).

But using (9.24)–(9.26) and Definition 7.1, and recalling Remark 9.11, we obtain, when $|z_0 + zz_2| > |zz_2| > 0$,

$$\langle e^{-(\log z)L'(0)} w'_{(4)}, (I_1^z \circ (1_{W_1} \otimes I_2^z))(w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)}) \rangle$$
$$= ((I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)})$$
$$= (\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}})(w_{(1)} \otimes w_{(2)}), \tag{9.58}$$

and for

$$z \in O \cap P \cap Q \cap \Pi \cap \Gamma,$$

the left-hand side equals (9.34). Thus for

$$z \in O \cap P \cap Q \cap \Pi \cap \Gamma$$

such that $|z_0 + zz_2| > |zz_2|$,

$$(\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}})(w_{(1)} \otimes w_{(2)})$$
$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}). \tag{9.59}$$

Since O, P, Q, Π and Γ are all independent of $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, (9.59) holds for $z \in O \cap P \cap Q \cap \Pi \cap \Gamma$ such that $|z_0 + zz_2| > |zz_2|$ and for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Thus for $z \in O \cap P \cap Q \cap \Pi \cap \Gamma$ such that $|z_0 + zz_2| > |zz_2|$, we have

$$\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}}$$
$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}), \tag{9.60}$$

where the right-hand side is understood as the sum of the weakly absolutely convergent (double) series denoted by the same notation (recall Remark 7.24).

Now I_1^z and I_2^z are $P(z_0 + zz_2)$- and $P(zz_2)$-intertwining maps, and when

$$|z_0 + zz_2| > |zz_2| > 0,$$

$I_1^z \circ (1_{W_1} \otimes I_2^z)$ is a $P(z_0 + zz_2, zz_2)$-intertwining map, by Proposition 8.5. Thus by Proposition 8.17,

$$(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^* \tag{9.61}$$

satisfies the $P(z_0 + zz_2, zz_2)$-compatibility condition, and

$$(z_0 + zz_2) - zz_2 = z_0.$$

Then by Lemma 9.3, when

$$|z_0 + zz_2| > |zz_2| > |z_0| > 0,$$

for $v \in V$ the coefficients of the monomials in x and x_1 in

$$x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) \left(Y'_{P(z_0)}(v, x) \mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}} \right) (w_{(1)} \otimes w_{(2)})$$

are absolutely convergent and we have

$$\left(\tau_{P(z_0)} \left(x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) Y_t(v, x) \right) \mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}} \right) (w_{(1)} \otimes w_{(2)})$$

$$= x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) \left(Y'_{P(z_0)}(v, x) \mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}} \right) (w_{(1)} \otimes w_{(2)}). \tag{9.62}$$

As we previewed in (9.26), let R be a sufficiently small open neighborhood of $z = 1$ such that

$$|z_0 + zz_2| > |zz_2| > |z_0| > 0 \tag{9.63}$$

for $z \in R$. Then for

$$z \in O \cap P \cap Q \cap \Pi \cap \Gamma \cap R,$$

(9.62) holds. From (9.60) and (9.62), we obtain

$$\left(\tau_{P(z_0)} \left(x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) Y_t(v, x) \right) \right.$$

$$\left. \left(\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) \right) \right) (w_{(1)} \otimes w_{(2)})$$

$$= x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) \cdot$$

$$\cdot \left(Y'_{P(z_0)}(v, x) \left(\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) \right) \right) (w_{(1)} \otimes w_{(2)}) \tag{9.64}$$

for

$$z \in O \cap P \cap Q \cap \Pi \cap \Gamma \cap R,$$

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a set that is independent of $w_{(1)}$ and $w_{(2)}$, and as in (9.62), the meaning of the right-hand side is that the coefficient of each monomial in x and x_1 is the sum of an absolutely convergent series, each term of which now involves the weakly absolutely convergent double sum over $n \in \mathbb{R}$ and $i = 0, \ldots, N$.

We shall need to bring the double sums over n and i to the outside, on both sides of (9.64).

First we do this for the left-hand side of (9.64). Using the definition (5.86) of $\tau_{P(z_0)}$ and the definition (2.57) of the opposite vertex operator Y^o, we can write the definition of $\tau_{P(z_0)}$ more explicitly as

$$
\left(\tau_{P(z_0)} \left(x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) Y_t(v, x) \right) \lambda \right) (w_{(1)} \otimes w_{(2)})
$$
$$
= z_0^{-1} \delta \left(\frac{x^{-1} - x_1}{z_0} \right) \lambda (Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_1) w_{(1)} \otimes w_{(2)})
$$
$$
+ x_1^{-1} \delta \left(\frac{z_0 - x^{-1}}{-x_1} \right) \lambda (w_{(1)} \otimes Y_2(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) w_{(2)}) \tag{9.65}
$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $\lambda \in (W_1 \otimes W_2)^*$. In particular, the left-hand side of (9.62) is equal to

$$
z_0^{-1} \delta \left(\frac{x^{-1} - x_1}{z_0} \right) \cdot
$$
$$
\cdot \mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}} (Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_1) w_{(1)} \otimes w_{(2)})
$$
$$
+ x_1^{-1} \delta \left(\frac{z_0 - x^{-1}}{-x_1} \right) \cdot
$$
$$
\cdot \mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_{(3)}} (w_{(1)} \otimes Y_2(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) w_{(2)}).
$$
$$
\tag{9.66}
$$

For
$$
z \in O \cap P \cap Q \cap \Pi \cap \Gamma \cap R,
$$
by (9.60), the coefficients of the monomials in x and x_1 in

$$
\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot
$$
$$
\cdot ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) (Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_1) w_{(1)} \otimes w_{(2)}) \tag{9.67}
$$

and

$$
\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot
$$
$$
\cdot ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) (w_{(1)} \otimes Y_2(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) w_{(2)})
$$
$$
\tag{9.68}
$$

are absolutely convergent to the corresponding coefficients of the monomials in x and x_1 in

$$\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}), e^{(\log z)L(0)}w_{(3)}} (Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_1)w_{(1)} \otimes w_{(2)})$$

and

$$\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}), e^{(\log z)L(0)}w_{(3)}} (w_{(1)} \otimes Y_2(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w_{(2)}),$$

respectively. Then, as finite linear combinations of the coefficients of the monomials in x and x_1 in (9.67) and (9.68), the coefficients of the monomials in x and x_1 in

$$z_0^{-1}\delta\left(\frac{x^{-1} - x_1}{z_0}\right)\left(\sum_{n \in \mathbb{R}}\sum_{i=0}^{N} e^{-n \log z}\frac{(-\log z)^i}{i!}\cdot\right.$$
$$\left.\cdot((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_1)w_{(1)} \otimes w_{(2)})\right)$$
$$+x_1^{-1}\delta\left(\frac{z_0 - x^{-1}}{-x_1}\right)\left(\sum_{n \in \mathbb{R}}\sum_{i=0}^{N} e^{-n \log z}\frac{(-\log z)^i}{i!}\cdot\right.$$
$$\left.\cdot((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes Y_2(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w_{(2)})\right)$$
$$\tag{9.69}$$

are absolutely convergent to the corresponding coefficients of the monomials in x and x_1 in (9.66).

Now for

$$z \in O \cap P \cap Q \cap \Pi \cap \Gamma \cap R,$$

we consider the coefficients of the monomials in x and x_1 in

$$\sum_{n \in \mathbb{R}}\sum_{i=0}^{N} e^{-n \log z}\frac{(-\log z)^i}{i!}\cdot$$
$$\cdot\left(\tau_{P(z_0)}\left(x_1^{-1}\delta\left(\frac{x^{-1} - z_0}{x_1}\right)Y_t(v, x)\right)((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})\right)$$
$$= \sum_{n \in \mathbb{R}}\sum_{i=0}^{N} e^{-n \log z}\frac{(-\log z)^i}{i!}z_0^{-1}\delta\left(\frac{x^{-1} - x_1}{z_0}\right)\cdot$$
$$\cdot((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_1)w_{(1)} \otimes w_{(2)})$$
$$+ \sum_{n \in \mathbb{R}}\sum_{i=0}^{N} e^{-n \log z}\frac{(-\log z)^i}{i!}x_1^{-1}\delta\left(\frac{z_0 - x^{-1}}{-x_1}\right)\cdot$$
$$\cdot((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)})(w_{(1)} \otimes Y_2(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w_{(2)}),$$
$$\tag{9.70}$$

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where we have used (9.65). The coefficient of each monomial in x and x_1 in the right-hand side of (9.70) is the sum over n and i of the sum of the corresponding monomial in x and x_1 in

$$e^{-n \log z} \frac{(-\log z)^i}{i!} z_0^{-1} \delta \left(\frac{x^{-1} - x_1}{z_0} \right) \cdot$$
$$\cdot ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) (Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_1)w_{(1)} \otimes w_{(2)}) \qquad (9.71)$$

and in

$$e^{-n \log z} \frac{(-\log z)^i}{i!} x_1^{-1} \delta \left(\frac{z_0 - x^{-1}}{-x_1} \right) \cdot$$
$$\cdot ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) (w_{(1)} \otimes Y_2(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w_{(2)}). \qquad (9.72)$$

Thus since finite linear combinations of the coefficients of the monomials in x and x_1 in (9.67) and (9.68) are absolutely convergent, the coefficients of the monomials in x and x_1 in the right-hand side of (9.70) are also absolutely convergent. Moreover, these (absolutely convergent) coefficients are equal to the (absolutely convergent) coefficients of the monomials in x and x_1 in (9.69). Thus by (9.70), for

$$z \in O \cap P \cap Q \cap \Pi \cap \Gamma \cap R,$$

the coefficient of each monomial in x and x_1 in

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot$$
$$\cdot \left(\tau_{P(z_0)} \left(x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) Y_t(v, x) \right) ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) (w_{(1)} \otimes w_{(2)}) \right)$$
$$(9.73)$$

is an absolutely convergent (double) series and converges to the coefficient of the corresponding monomial in x and x_1 in the left-hand side of (9.62) (or of (9.64)).

Now we need to bring the double sum over n and i on the right-hand side of (9.64) to the outside, and in the process, we shall need to increase N and restrict the range of z.

Taking Res_{x_1} in (9.73) and using (5.21), we see that the coefficient of each monomial in x in

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} Y'_{P(z_0)}(v, x) ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) (w_{(1)} \otimes w_{(2)}) \qquad (9.74)$$

is an absolutely convergent series and that it converges to the coefficient of the corresponding monomial in x in the result of applying Res_{x_1} to the left-hand side of (9.62) (or of (9.64)), namely,

$$\left(Y'_{P(z_0)}(v, x) \left(\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}) \right) \right) (w_{(1)} \otimes w_{(2)}) \qquad (9.75)$$

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(recall (9.60)).

By (9.1), with the continuing assumption on z, we thus have

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{-n\log z}\frac{(-\log z)^i}{i!}Y'_{P(z_0)}(v,x)((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)})(w_{(1)}\otimes w_{(2)})$$

$$+\mathrm{Res}_{x_0^{-1}}x\delta\Big(\frac{-zz_2+x_0^{-1}}{x^{-1}}\Big)((I_1^z\circ(1_{W_1}\otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}))(w_{(1)}\otimes w_{(2)}$$

$$\otimes Y_3(e^{xL(1)}(-x^{-2})^{L(0)}v,x_0^{-1})e^{(\log z)L(0)}w_{(3)})$$

$$=\Big(Y'_{P(z_0)}(v,x)\Big(\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{-n\log z}\frac{(-\log z)^i}{i!}((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)})\Big)\Big)(w_{(1)}\otimes w_{(2)})$$

$$+\mathrm{Res}_{x_0^{-1}}x\delta\Big(\frac{-zz_2+x_0^{-1}}{x^{-1}}\Big)((I_1^z\circ(1_{W_1}\otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}))(w_{(1)}\otimes w_{(2)}$$

$$\otimes Y_3(e^{xL(1)}(-x^{-2})^{L(0)}v,x_0^{-1})e^{(\log z)L(0)}w_{(3)})$$

$$=(Y'_{P(z_0)}(v,x)\mu^{(2)}_{(I_1^z\circ(1_{W_1}\otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}),e^{(\log z)L(0)}w_{(3)}})(w_{(1)}\otimes w_{(2)})$$

$$+\mathrm{Res}_{x_0^{-1}}x\delta\Big(\frac{-zz_2+x_0^{-1}}{x^{-1}}\Big)((I_1^z\circ(1_{W_1}\otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}))(w_{(1)}\otimes w_{(2)}$$

$$\otimes Y_3(e^{xL(1)}(-x^{-2})^{L(0)}v,x_0^{-1})e^{(\log z)L(0)}w_{(3)})$$

$$=\mathrm{Res}_{x_0^{-1}}\Big(\tau_{P(z_0+zz_2,zz_2)}\Big(x\delta\Big(\frac{x_0^{-1}-zz_2}{x^{-1}}\Big)Y_t((-x_0^2)^{L(0)}e^{-x_0L(1)}e^{xL(1)}(-x^{-2})^{L(0)}v,x_0)\Big)\cdot$$

$$\cdot((I_1^z\circ(1_{W_1}\otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}))\Big)(w_{(1)}\otimes w_{(2)}\otimes e^{(\log z)L(0)}w_{(3)}),$$

$$(9.76)$$

where the equalities include the information that the coefficient of each monomial in each expression involving a double sum is absolutely convergent.

Recall from the $P^{(2)}(z_0)$-grading condition that the elements $\lambda_n^{(2)}$ for $n\in\mathbb{R}$ depend on $w_{(3)}$ (and on $w'_{(4)}$) and that the sets Π and Γ cannot be assumed independent of $w_{(3)}$, nor can our integer $N\in\mathbb{N}$. Since we now need to use $\lambda_n^{(2)}$, $n\in\mathbb{R}$, for different $w_{(3)}$, we denote $\lambda_n^{(2)}$ by $\lambda_n^{(2)}[w_{(3)}]$ for $n\in\mathbb{R}$ and we denote Π, Γ and N by $\Pi[w_{(3)}]$, $\Gamma[w_{(3)}]$ and $N[w_{(3)}]$, respectively. Then with these notations, the convergence properties and formulas that we have proved hold for all $w_{(3)}\in W_3$.

In order to handle the second term in the left-hand side of (9.76), we shall need to consider the following application of a certain conjugated operator to $w_{(3)}$, and to treat this element as an analogue of $w_{(3)}$:

$$X = e^{-(\log z)L(0)}Y_3(e^{xL(1)}(-x^{-2})^{L(0)}v,x_0^{-1})e^{(\log z)L(0)}w_{(3)}$$

$$= Y_3(e^{-(\log z)L(0)}e^{xL(1)}(-x^{-2})^{L(0)}v,e^{-\log z}x_0^{-1})w_{(3)}, \qquad (9.77)$$

where we obtain the second expression by using the conjugation formula (3.86). Since

$e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$ and $Y_3(u, y)w_{(3)} \in W_3((y))$ for $u \in V$ and y a formal variable, the right-hand side of (9.77) is of the form

$$Y_3(z^{-L(0)}e^{xL(1)}(-x^{-2})^{L(0)}v, (zx_0)^{-1})w_{(3)} = \sum_{l \leq L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} z^{s+l} w_s^{l,m} x_0^l x^m$$

for certain integers L, $M \geq 0$ and $S \geq 0$ and certain (determined) elements $w_s^{l,m} \in W_3$. That is,

$$X = \sum_{l \leq L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} z^{s+l} w_s^{l,m} x_0^l x^m.$$

With this notation, for $z \in R$ (recall (9.63) and (9.58)), the second term in the left-hand side of (9.76) equals

$$\operatorname{Res}_{x_0^{-1}} x \delta\left(\frac{-zz_2 + x_0^{-1}}{x^{-1}}\right) Z, \tag{9.78}$$

where

$$Z = (I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} X). \tag{9.79}$$

The coefficient of $x_0^l x^m$ in (9.79) is the precisely determined expression

$$(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)})) \left(w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} \left(\sum_{s=-S}^{S} z^{s+l} w_s^{l,m} \right) \right)$$

$$= \sum_{s=-S}^{S} z^{s+l} (I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_s^{l,m}),$$

which by (9.58) equals

$$\sum_{s=-S}^{S} z^{s+l} \left(\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_s^{l,m}} \right) (w_{(1)} \otimes w_{(2)}) \tag{9.80}$$

(with $z \in R$, as we have assumed above). In particular,

$$Z = \sum_{l \leq L} \sum_{m=-M}^{M} \left(\sum_{s=-S}^{S} z^{s+l} \left(\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)}), e^{(\log z)L(0)} w_s^{l,m}} \right) (w_{(1)} \otimes w_{(2)}) \right) x_0^l x^m. \tag{9.81}$$

Now the residue (9.78) involves only finitely many powers of x_0 in (9.81), and in particular, the coefficient of each monomial in x in (9.78) is a finite linear combination of expressions of the form (9.80), involving only finitely many l, independently of the power of x in (9.78). We apply our results above to the corresponding finite family of elements $w_s^{l,m}$ that arise in this way, and we use $\lambda_n^{(2)}[w_s^{l,m}]$, $\Pi[w_s^{l,m}]$, $\Gamma[w_s^{l,m}]$ and $N[w_s^{l,m}]$, as defined above. Let

$$\widetilde{N} = \max\{N[w_s^{l,m}]\},$$

$$\widetilde{\Pi} = \bigcap(\Pi[w_s^{l,m}]),$$

$$\widetilde{\Gamma} = \bigcap(\Gamma[w_s^{l,m}]),$$

and let

$$z \in O \cap P \cap Q \cap \widetilde{\Pi} \cap \widetilde{\Gamma} \cap R. \tag{9.82}$$

Then by (9.60),

$$\mu^{(2)}_{(I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)}w'_{(4)}),e^{(\log z)L(0)}w_s^{l,m}}$$

$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}[w_s^{l,m}]), \tag{9.83}$$

where the right-hand side is understood as the sum of a weakly absolutely convergent double series. For $n \in \mathbb{R}$, form the formal Laurent series

$$\Lambda_n^{(2)}(x_0, x; z) = \sum_{l \leq L} \sum_{m=-M}^{M} \left(\sum_{s=-S}^{S} z^{s+l} \lambda_n^{(2)}[w_s^{l,m}] \right) x_0^l x^m. \tag{9.84}$$

Then the coefficient of each power of x in the second term in the left-hand side of (9.76) equals its coefficient in

$$\text{Res}_{x_0^{-1}} x \delta \left(\frac{-zz_2 + x_0^{-1}}{x^{-1}} \right) Z$$

$$= \text{Res}_{x_0^{-1}} x \delta \left(\frac{-zz_2 + x_0^{-1}}{x^{-1}} \right) \sum_{l \leq L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} z^{s+l} \cdot$$

$$\cdot \left(\sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}[w_s^{l,m}])(w_{(1)} \otimes w_{(2)}) \right) x_0^l x^m$$

$$= \text{Res}_{x_0^{-1}} x \delta \left(\frac{-zz_2 + x_0^{-1}}{x^{-1}} \right) \cdot$$

$$\cdot \sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} ((L'_{P(z_0)}(0) - n)^i \Lambda_n^{(2)}(x_0, x; z))(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} \text{Res}_{x_0^{-1}} x \delta \left(\frac{-zz_2 + x_0^{-1}}{x^{-1}} \right) \cdot$$

$$\cdot ((L'_{P(z_0)}(0) - n)^i \Lambda_n^{(2)}(x_0, x; z))(w_{(1)} \otimes w_{(2)}), \tag{9.85}$$

where we have double absolute convergence; recall that $\widetilde{N}$ and the range (9.82) of z are independent of the power of x.

In order to reach our goal of bringing the double sum over n and i to the outside on the right-hand side of (9.64), we need to multiply (9.74), (9.75) and the other expressions in (9.76) by

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big);\tag{9.86}$$

in particular, we need to show that we can do this for each of the expressions.

First we do this for the expressions in (9.85), which are equal to the second term in the left-hand side of (9.76). Since $|zz_2| > |z_0|$, by Lemma 8.1, formula (8.7),

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)x\delta\Big(\frac{-zz_2+x_0^{-1}}{x^{-1}}\Big)\tag{9.87}$$

is a formal Laurent series in x, x_1 and x_0 each of whose coefficients is an absolutely convergent series of the form $\sum_{j\in\mathbb{N}} a_j$ $(a_j \in \mathbb{C})$. Thus for

$$z \in O \cap P \cap Q \cap \widetilde{\Pi} \cap \widetilde{\Gamma} \cap R,$$

the coefficient of each monomial in x and x_1 in

$$
x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}}e^{-n\log z}\frac{(-\log z)^i}{i!}\mathrm{Res}_{x_0^{-1}}x\delta\Big(\frac{-zz_2+x_0^{-1}}{x^{-1}}\Big)\cdot
$$
$$
\cdot\big((L'_{P(z_0)}(0)-n)^i\Lambda_n^{(2)}(x_0,x;z)\big)\big(w_{(1)}\otimes w_{(2)}\big)
$$
$$
= x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\mathrm{Res}_{x_0^{-1}}x\delta\Big(\frac{-zz_2+x_0^{-1}}{x^{-1}}\Big)\cdot
$$
$$
\cdot\left(\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}}e^{-n\log z}\frac{(-\log z)^i}{i!}\big((L'_{P(z_0)}(0)-n)^i\Lambda_n^{(2)}(x_0,x;z)\big)\big(w_{(1)}\otimes w_{(2)}\big)\right)
$$
$$
= \mathrm{Res}_{x_0^{-1}}x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)x\delta\Big(\frac{-zz_2+x_0^{-1}}{x^{-1}}\Big)\cdot
$$
$$
\cdot\left(\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}}e^{-n\log z}\frac{(-\log z)^i}{i!}\big((L'_{P(z_0)}(0)-n)^i\Lambda_n^{(2)}(x_0,x;z)\big)\big(w_{(1)}\otimes w_{(2)}\big)\right)
$$
$$\tag{9.88}$$

is a finite linear combination of products of pairs of absolutely convergent series and hence is a finite linear combination of absolutely convergent triple series (over $j \in \mathbb{N}$, $n \in \mathbb{R}$ and $i = 0, \ldots, \widetilde{N}$). In particular, the second term in the left-hand side of (9.76) can be multiplied by (9.86), in the sense of absolute convergence, and moreover, for

$$z \in O \cap P \cap Q \cap \widetilde{\Pi} \cap \widetilde{\Gamma} \cap R,$$

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the coefficient of each monomial in x and x_1 in

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}} e^{-n\log z}\frac{(-\log z)^i}{i!}\cdot$$

$$\cdot\mathrm{Res}_{x_0^{-1}} x_1^{-1}\delta\left(\frac{x^{-1}-z_0}{x_1}\right)x\delta\left(\frac{-zz_2+x_0^{-1}}{x^{-1}}\right)((L'_{P(z_0)}(0)-n)^i\Lambda_n^{(2)}(x_0,x;z))(w_{(1)}\otimes w_{(2)})$$

$$(9.89)$$

is absolutely convergent to the coefficient of the corresponding monomial in (9.88).

We now re-express (9.88) and (9.89) by using the explicit dependence of (9.84) on z. Since the coefficient of each power of x_0 in (9.84) is a finite sum, we have

$$x_1^{-1}\delta\left(\frac{x^{-1}-z_0}{x_1}\right)x\delta\left(\frac{-zz_2+x_0^{-1}}{x^{-1}}\right)((L'_{P(z_0)}(0)-n)^i\Lambda_n^{(2)}(x_0,x;z))(w_{(1)}\otimes w_{(2)})$$

$$= x_1^{-1}\delta\left(\frac{x^{-1}-z_0}{x_1}\right)x\delta\left(\frac{-zz_2+x_0^{-1}}{x^{-1}}\right)\cdot$$

$$\cdot\sum_{l\leq L}\sum_{m=-M}^{M}\sum_{s=-S}^{S} z^{s+l}((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)}[w_s^{l,m}])(w_{(1)}\otimes w_{(2)})x_0^l x^m$$

$$= \sum_{l\leq L}\sum_{m=-M}^{M}\sum_{s=-S}^{S} z^{s+l} x_1^{-1}\delta\left(\frac{x^{-1}-z_0}{x_1}\right)x\delta\left(\frac{-zz_2+x_0^{-1}}{x^{-1}}\right)\cdot$$

$$\cdot((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)}[w_s^{l,m}])(w_{(1)}\otimes w_{(2)})x_0^l x^m.$$

Thus for

$$z\in O\cap P\cap Q\cap\widetilde{\Pi}\cap\widetilde{\Gamma}\cap R,$$

the coefficient of each monomial in x and x_1 in (9.89), written as

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}} e^{-n\log z}\frac{(-\log z)^i}{i!}\cdot$$

$$\cdot\mathrm{Res}_{x_0^{-1}}\sum_{l\leq L}\sum_{m=-M}^{M}\sum_{s=-S}^{S} z^{s+l} x_1^{-1}\delta\left(\frac{x^{-1}-z_0}{x_1}\right)x\delta\left(\frac{-zz_2+x_0^{-1}}{x^{-1}}\right)\cdot$$

$$\cdot((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)}[w_s^{l,m}])(w_{(1)}\otimes w_{(2)})x_0^l x^m$$

$$= \sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}}\sum_{l\leq L}\sum_{m=-M}^{M}\sum_{s=-S}^{S} e^{-(n-s-l)\log z}\frac{(-\log z)^i}{i!}\cdot$$

$$\cdot\mathrm{Res}_{x_0^{-1}} x_1^{-1}\delta\left(\frac{x^{-1}-z_0}{x_1}\right)x\delta\left(\frac{-zz_2+x_0^{-1}}{x^{-1}}\right)\cdot$$

$$\cdot((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)}[w_s^{l,m}])(w_{(1)}\otimes w_{(2)})x_0^l x^m$$

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$$= \sum_{p \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} \sum_{l \leq L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} e^{-p \log z} \frac{(-\log z)^i}{i!} \cdot$$

$$\cdot \mathrm{Res}_{x_0^{-1}} x_1^{-1} \delta\Big(\frac{x^{-1} - z_0}{x_1} \Big) x \delta\Big(\frac{-z z_2 + x_0^{-1}}{x^{-1}} \Big) \cdot$$

$$\cdot ((L'_{P(z_0)}(0) - (p + s + l))^i \lambda^{(2)}_{p+s+l}[w_s^{l,m}])(w_{(1)} \otimes w_{(2)}) x_0^l x^m$$

$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot$$

$$\cdot \mathrm{Res}_{x_0^{-1}} x_1^{-1} \delta\Big(\frac{x^{-1} - z_0}{x_1} \Big) x \delta\Big(\frac{-z z_2 + x_0^{-1}}{x^{-1}} \Big) \cdot$$

$$\cdot \sum_{l \leq L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} ((L'_{P(z_0)}(0) - (n + s + l))^i \lambda^{(2)}_{n+s+l}[w_s^{l,m}])(w_{(1)} \otimes w_{(2)}) x_0^l x^m$$

$$(9.90)$$

is absolutely convergent to the coefficient of the corresponding monomial in (9.88), written as

$$x_1^{-1} \delta\Big(\frac{x^{-1} - z_0}{x_1} \Big) \mathrm{Res}_{x_0^{-1}} x \delta\Big(\frac{-z z_2 + x_0^{-1}}{x^{-1}} \Big) \cdot$$

$$\cdot \sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot$$

$$\cdot \sum_{l \leq L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} z^{s+l} ((L'_{P(z_0)}(0) - n)^i \lambda^{(2)}_n[w_s^{l,m}])(w_{(1)} \otimes w_{(2)}) x_0^l x^m$$

$$= x_1^{-1} \delta\Big(\frac{x^{-1} - z_0}{x_1} \Big) \mathrm{Res}_{x_0^{-1}} x \delta\Big(\frac{-z z_2 + x_0^{-1}}{x^{-1}} \Big) \cdot$$

$$\cdot \sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot$$

$$\cdot \sum_{l \leq L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} ((L'_{P(z_0)}(0) - (n + s + l))^i \lambda^{(2)}_{n+s+l}[w_s^{l,m}])(w_{(1)} \otimes w_{(2)}) x_0^l x^m.$$

$$(9.91)$$

Moreover, recall that (9.88), and hence (9.91), is equal to the product, in the sense of absolute convergence, of (9.86) and the second term in the left-hand side of (9.76).

Next we show that we can multiply the right-hand side of (9.76) by (9.86), by using the $P(z_0 + z z_2, z z_2)$-compatibility condition. Since the expression (9.61) satisfies this condition, we have

$$\Big(\tau_{P(z_0+z z_2, z z_2)} \Big(x_1^{-1} \delta\Big(\frac{x_0^{-1} - (z_0 + z z_2)}{x_1} \Big) x \delta\Big(\frac{x_0^{-1} - z z_2}{x^{-1}} \Big) Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x L(1)} (-x^{-2})^{L(0)} v, x_0) \Big) \Big) \cdot$$

$$\cdot \big((I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)})) \big) (w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)})$$

$$= x_1^{-1} \delta\Big(\frac{x_0^{-1} - (z_0 + zz_2)}{x_1} \Big) x \delta\Big(\frac{x_0^{-1} - zz_2}{x^{-1}} \Big) \Big(Y'_{P(z_0 + zz_2, zz_2)}((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x L(1)} (-x^{-2})^{L(0)} v, x_0) \cdot$$

$$\cdot \big((I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)})) \big) (w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)}),$$

and taking Res_{x_1} and using Remark 8.18 (and in particular, (8.50)), and then applying $\mathrm{Res}_{x_0^{-1}}$, we obtain

$$\mathrm{Res}_{x_0^{-1}} \bigg(\tau_{P(z_0 + zz_2, zz_2)} \Big(x\delta\Big(\frac{x_0^{-1} - zz_2}{x^{-1}} \Big) Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x L(1)} (-x^{-2})^{L(0)} v, x_0) \Big) \cdot$$

$$\cdot \big((I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)})) \big) (w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)})$$

$$= \mathrm{Res}_{x_0^{-1}} x\delta\Big(\frac{x_0^{-1} - zz_2}{x^{-1}} \Big) \Big(Y'_{P(z_0 + zz_2, zz_2)}((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x L(1)} (-x^{-2})^{L(0)} v, x_0) \cdot$$

$$\cdot \big((I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)})) \big) (w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)})$$

$$= \mathrm{Res}_{x_0^{-1}} x_0 \delta\Big(\frac{x^{-1} + zz_2}{x_0^{-1}} \Big) \Big(Y'_{P(z_0 + zz_2, zz_2)}((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x L(1)} (-x^{-2})^{L(0)} v, x_0) \cdot$$

$$\cdot \big((I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)})) \big) (w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)}).$$
$$\tag{9.92}$$

The right-hand side of (9.92) and thus the left-hand side, which is the right-hand side of (9.76), involves only finitely many negative powers of x. In particular, the sum of the two terms on the left-hand side of (9.76) involves only finitely many negative powers of x and hence lies in $x^{m_0} \mathbb{C}[[x]]$ for some $m_0 \in \mathbb{Z}$, so that the coefficients of x^m for $m < m_0$ in the left-hand side of (9.76) cancel. Moreover, we can multiply the right-hand side of (9.76) by (9.86), to obtain

$$x_1^{-1} \delta\Big(\frac{x^{-1} - z_0}{x_1} \Big) \mathrm{Res}_{x_0^{-1}} \bigg(\tau_{P(z_0 + zz_2, zz_2)} \Big(x\delta\Big(\frac{x_0^{-1} - zz_2}{x^{-1}} \Big) Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x L(1)} (-x^{-2})^{L(0)} v, x_0) \Big) \cdot$$

$$\cdot \big((I_1^z \circ (1_{W_1} \otimes I_2^z))'(e^{-(\log z)L'(0)} w'_{(4)})) \big) (w_{(1)} \otimes w_{(2)} \otimes e^{(\log z)L(0)} w_{(3)}).$$
$$\tag{9.93}$$

The first term in the left-hand side of (9.76) is equal to

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} g_{n,i}(x),$$

where

$$g_{n,i}(x) = Y'_{P(z_0)}(v, x)((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}[w_{(3)}])(w_{(1)} \otimes w_{(2)}) \tag{9.94}$$

for $n \in \mathbb{R}$ and $i = 0, \dots, N$, where we have double absolute convergence for

$$z \in O \cap P \cap Q \cap \Pi[w_{(3)}] \cap \Gamma[w_{(3)}] \cap R.$$

By (9.84) and (9.85), which we rewrite as in (9.91), the second term in the left-hand side of (9.76) is equal to

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} h_{n,i}(x), \tag{9.95}$$

where

$$
\begin{aligned}
h_{n,i}(x) \quad = \quad & \operatorname{Res}_{x_0^{-1}} x \delta\left(\frac{-z z_2 + x_0^{-1}}{x^{-1}}\right) \cdot \\
& \cdot \sum_{l \le L} \sum_{m=-M}^{M} \sum_{s=-S}^{S} ((L'_{P(z_0)}(0) - (n+s+l))^i \lambda_{n+s+l}^{(2)}[w_s^{l,m}])(w_{(1)} \otimes w_{(2)}) x_0^l x^m
\end{aligned}
\tag{9.96}
$$

for $n \in \mathbb{R}$ and $i = 0, \dots, \widetilde{N}$; here we have double absolute convergence for

$$z \in O \cap P \cap Q \cap \widetilde{\Pi} \cap \widetilde{\Gamma} \cap R.$$

For $N < i \le \max(N, \widetilde{N})$, set $g_{n,i}(x) = 0$ and for $\widetilde{N} < i \le \max(N, \widetilde{N})$, set $h_{n,i}(x) = 0$. Let

$$f_{n,i}(x) = g_{n,i}(x) + h_{n,i}(x)$$

for $n \in \mathbb{R}$ and $i = 0, \dots, \max(N, \widetilde{N})$. Then the left-hand side of (9.76) is equal to

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{\max(N,\widetilde{N})} e^{-n \log z} \frac{(-\log z)^i}{i!} f_{n,i}(x), \tag{9.97}$$

with double absolute convergence for each power of x for

$$z \in O \cap P \cap Q \cap \Pi[w_{(3)}] \cap \widetilde{\Pi} \cap \Gamma[w_{(3)}] \cap \widetilde{\Gamma} \cap R.$$

That is, for each $m \in \mathbb{Z}$,

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{\widetilde{N}} e^{-n \log z} \frac{(-\log z)^i}{i!} \operatorname{Res}_x x^{-m-1} f_{n,i}(x) \tag{9.98}$$

is absolutely convergent for z in this set.

Recall that the left-hand side of (9.76) in fact lies in $x^{m_0}\mathbb{C}[[x]]$. Thus for $m < m_0$,

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}} e^{-n\log z}\frac{(-\log z)^i}{i!}\mathrm{Res}_x x^{-m-1}f_{n,i}(x) = 0$$

for

$$z \in O \cap P \cap Q \cap \Pi[w_{(3)}] \cap \widetilde{\Pi} \cap \Gamma[w_{(3)}] \cap \widetilde{\Gamma} \cap R.$$

Since by Proposition 7.8 $\mathbb{R} \times \{0,\dots,\max(N,\widetilde{N})\}$ is a unique expansion set,

$$\mathrm{Res}_x x^{-m-1}f_{n,i}(x) = 0$$

for $m < m_0$, $n \in \mathbb{R}$ and $i = 0,\dots,\max(N,\widetilde{N})$, and so

$$f_{n,i}(x) \in x^{m_0}\mathbb{C}[[x]]$$

for $n \in \mathbb{R}$ and $i = 0,\dots,\max(N,\widetilde{N})$. Thus for

$$z \in O \cap P \cap Q \cap \Pi[w_{(3)}] \cap \widetilde{\Pi} \cap \Gamma[w_{(3)}] \cap \widetilde{\Gamma} \cap R,$$

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{\max(N,\widetilde{N})} e^{-n\log z}\frac{(-\log z)^i}{i!}x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)f_{n,i}(x) \qquad (9.99)$$

is a formal Laurent series in x and x_1 whose coefficients, as finite linear combinations of the coefficients of powers of x in (9.97), are equal to the corresponding (absolutely convergent) coefficients in

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\sum_{n\in\mathbb{R}}\sum_{i=0}^{\max(N,\widetilde{N})} e^{-n\log z}\frac{(-\log z)^i}{i!}f_{n,i}(x).$$

This expression equals the product of (9.86) and the left-hand side of (9.76), and hence (9.99) is absolutely convergent to (9.93).

Moreover, from (9.88)–(9.91), (9.97) and (9.98), for

$$z \in O \cap P \cap Q \cap \widetilde{\Pi} \cap \widetilde{\Gamma} \cap R,$$

the coefficient of each monomial in x and x_1 in

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}} e^{-n\log z}\frac{(-\log z)^i}{i!}x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)h_{n,i}(x)$$

$$= \sum_{n\in\mathbb{R}}\sum_{i=0}^{\max(N,\widetilde{N})} e^{-n\log z}\frac{(-\log z)^i}{i!}x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)h_{n,i}(x)$$

is absolutely convergent to the corresponding coefficient in

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\sum_{n\in\mathbb{R}}\sum_{i=0}^{\widetilde{N}}e^{-n\log z}\frac{(-\log z)^i}{i!}h_{n,i}(x)$$

$$=x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\sum_{n\in\mathbb{R}}\sum_{i=0}^{\max(N,\widetilde{N})}e^{-n\log z}\frac{(-\log z)^i}{i!}h_{n,i}(x).$$

Assume that

$$z\in O\cap P\cap Q\cap\Pi[w_{(3)}]\cap\widetilde{\Pi}\cap\Gamma[w_{(3)}]\cap\widetilde{\Gamma}\cap R.$$

We are now ready to bring the double sum over n and i in the right-hand side of (9.64) to the outside. This right-hand side equals

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\cdot$$

$$\cdot\Big(Y'_{P(z_0)}(v,x)\Big(\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{-n\log z}\frac{(-\log z)^i}{i!}((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)})[w_{(3)}]\Big)\Big)(w_{(1)}\otimes w_{(2)});$$

(9.100)

we recall that the coefficient of each monomial in x and x_1 in (9.100) is the sum of an absolutely convergent series, each term of which involves the weakly absolutely convergent double sum over n and i. Using the absolute convergence of the coefficients in (9.74) to those in (9.75), we rewrite (9.100) as

$$x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{-n\log z}\frac{(-\log z)^i}{i!}g_{n,i}(x).$$

(9.101)

What we need to show is that the coefficient of each monomial in x and x_1 in

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{-n\log z}\frac{(-\log z)^i}{i!}\cdot$$

$$\cdot x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)\Big(Y'_{P(z_0)}(v,x)((L'_{P(z_0)}(0)-n)^i\lambda_n^{(2)}[w_{(3)}])\Big)(w_{(1)}\otimes w_{(2)})$$

$$=\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{-n\log z}\frac{(-\log z)^i}{i!}x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)g_{n,i}(x)$$

(9.102)

is absolutely convergent and that it converges to the corresponding (absolutely convergent) coefficient in (9.101). We have:

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{-n\log z}\frac{(-\log z)^i}{i!}x_1^{-1}\delta\Big(\frac{x^{-1}-z_0}{x_1}\Big)g_{n,i}(x)$$

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$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{\max(N,\widetilde{N})} e^{-n \log z} \frac{(-\log z)^i}{i!} x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) g_{n,i}(x)$$

$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{\max(N,\widetilde{N})} e^{-n \log z} \frac{(-\log z)^i}{i!} x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) f_{n,i}(x)$$

$$- \sum_{n \in \mathbb{R}} \sum_{i=0}^{\max(N,\widetilde{N})} e^{-n \log z} \frac{(-\log z)^i}{i!} x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) h_{n,i}(x),$$

and the coefficient of each monomial in x and x_1 is absolutely convergent to the corresponding coefficient in

$$x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) \sum_{n \in \mathbb{R}} \sum_{i=0}^{\max(N,\widetilde{N})} e^{-n \log z} \frac{(-\log z)^i}{i!} f_{n,i}(x)$$

$$- x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) \sum_{n \in \mathbb{R}} \sum_{i=0}^{\max(N,\widetilde{N})} e^{-n \log z} \frac{(-\log z)^i}{i!} h_{n,i}(x)$$

$$= x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) \sum_{n \in \mathbb{R}} \sum_{i=0}^{\max(N,\widetilde{N})} e^{-n \log z} \frac{(-\log z)^i}{i!} g_{n,i}(x)$$

$$= x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) \sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} g_{n,i}(x),$$

as desired. (Note that in particular, the coefficient of each monomial in the inner expression $x_1^{-1}\delta\left(\frac{x^{-1}-z_0}{x_1}\right) g_{n,i}(x)$ in (9.102) is absolutely convergent, since it is essentially a sub-sum of the relevant absolutely convergent series.)

We have succeeded in bringing the double sums on both sides of (9.64) to the outside: For

$$z \in O \cap P \cap Q \cap \Pi[w_{(3)}] \cap \widetilde{\Pi} \cap \Gamma[w_{(3)}] \cap \widetilde{\Gamma} \cap R,$$

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot$$

$$\cdot \left(\tau_{P(z_0)} \left(x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) Y_t(v,x) \right) \left((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}[w_{(3)}] \right) \right) (w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{N} e^{-n \log z} \frac{(-\log z)^i}{i!} \cdot$$

$$\cdot x_1^{-1} \delta\left(\frac{x^{-1} - z_0}{x_1}\right) \left(Y'_{P(z_0)}(v,x) \left((L'_{P(z_0)}(0) - n)^i \lambda_n^{(2)}[w_{(3)}] \right) \right) (w_{(1)} \otimes w_{(2)}), \quad (9.103)$$

where the coefficients of the monomials in x and x_1 in the double sums on both sides of (9.103) are absolutely convergent and are equal.

Now we are able to apply Proposition 7.8. Since $\mathbb{R} \times \{0, \ldots, N\}$ is a unique expansion set, we conclude that the expansion coefficients of the double sums on the left- and right-hand sides of (9.103) are equal. In particular, taking $i = 0$, we obtain

$$
\left(\tau_{P(z_0)} \left(x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) Y_t(v, x) \right) \lambda_n^{(2)}[w_{(3)}] \right) (w_{(1)} \otimes w_{(2)})
$$
$$
= x_1^{-1} \delta \left(\frac{x^{-1} - z_0}{x_1} \right) (Y'_{P(z_0)}(v, x) \lambda_n^{(2)}[w_{(3)}]) (w_{(1)} \otimes w_{(2)})
$$

for each $n \in \mathbb{R}$, and so we have proved that each $\lambda_n^{(2)} = \lambda_n^{(2)}[w_{(3)}]$ satisfies Part (b) of the $P(z_0)$-compatibility condition.

This completes the proof. $\square$

Remark 9.18 Here we relate the proof of Theorem 9.17 to the corresponding analysis for the special case treated in [H]. The proof above of Part (b) of the $P(z_0)$-compatibility condition is a generalization of the proof of (14.51) in [H]. In the proof of (14.51) in [H], for a series of the form

$$
\sum_{n \in D} a_n z^{-n} \tag{9.104}
$$

where D is a strictly increasing sequence in $\mathbb{R}$, in order to determine the coefficients $a_n \in \mathbb{C}$ uniquely from the sum of the series, the series is required to be absolutely convergent in an open set of the form $0 < |z^{-1}| < r$ because Lemma 14.5 in [H] was proved in [H] only for such a series. (Here the first author would like to correct some minor mistakes in [H]: First, in the $P(z_2)$-local grading-restriction condition (respectively, in the $P(z_1 - z_2)$-local grading-restriction condition) in [H], we should require that the series depending on z' obtained by applying $e^{z' L'_{P(z_2)}(0)}$ (respectively, $e^{z' L'_{P(z_1 - z_2)}(0)}$) to each term of the weakly absolutely convergent series of $P(z_2)$-weight vectors in $(W_2 \otimes W_3)^*$ (respectively, in $(W_1 \otimes W_2)^*$) be weakly absolutely convergent for z' in a neighborhood of $z' = 0$. This is implicitly used in the proof of (14.51) in [H] and follows easily from the convergence and extension property in [H]. But it is not clear to the first author whether this can be proved by assuming the $P(z_2)$-local grading-restriction condition or the $P(z_1 - z_2)$-local grading-restriction condition in [H] for all z_1 and z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Second, in the proof of (14.51) in [H], the domain

$$
0 < |z| < \frac{|z_0|}{2|z_2|}
$$

should be replaced by the intersection of

$$
0 < |z^{-1}| < \frac{|z_2|}{|z_0|} \quad (> 1)
$$

and

$$|z_0 + zz_2| > |zz_2| > 0;$$

since the expansion (14.48) and consequently the series in (14.49) is of the form (9.104) rather than $\sum_{n \in D} a_n z^n$, the correct domain is

$$0 < |z^{-1}| < \frac{|z_2|}{|z_0|}.$$

The reason why the right-hand side of (14.48) is absolutely convergent in this domain and not just in its intersection with

$$|z_0 + zz_2| > |zz_2| > 0$$

is that when D is a strictly increasing sequence in $\mathbb{R}$, the absolute convergence of (9.104) at one particular z such that

$$|z^{-1}| = r \neq 0$$

implies that it is also absolutely convergent at any z satisfying

$$0 < |z^{-1}| \leq r.)$$

However, in our proof of Theorem 9.17 above, because $\mathbb{R} \times \{0, \ldots, N\}$ is a unique expansion set, the double absolute convergence of

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} a_{n,i} z^{-n} (- \log z)^i$$

for z in *any* nonempty open subset of $\mathbb{C}^{\times}$, not necessarily containing an open subset of the form $0 < |z^{-1}| < r$, implies that the coefficients $a_{n,i}$ are uniquely determined by the sum of the series; here we do not need the absolute convergence of the series for z^{-1} near 0. But our convergence-condition assumption gives only the absolute convergence of *iterated* series of the form

$$\sum_{n \in \mathbb{R}} \left(\sum_{i=0}^{N} a_{n,i} z^{-n} (- \log z)^i \right)$$

in a nonempty open set, and we had to prove the absolute convergence of the corresponding double series

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N} a_{n,i} z^{-n} (- \log z)^i$$

in the same open set using Proposition 7.9 (or Corollary 7.10). This proof of the absolute convergence of these double series is one of the hard parts of the proof of Theorem 9.17, and this was not needed in the proof of (14.51) in [H] because there we have no finite sum over powers of $\log z$. Another hard part of the proof above is to show that (9.64) implies that each $\lambda_n^{(2)}$ satisfies the $P(z_0)$-compatibility condition. This part of the proof amounts to

a proof that certain triple series are absolutely convergent, so that suitable iterated sums exist and are equal. (In fact, this part of the proof was also needed in the proof of (14.51) in [H] (with double rather than triple sums, since there are no finite sums over powers of $\log z$) but was not given there. The last part of the proof of Theorem 9.17 above gives this missing detail, in our present much more general case.) Also, even in the case considered in [H], the proof of Theorem 9.17 above establishes a stronger statement than (14.51) in [H]: For each $n \in \mathbb{R}$, $\lambda_n^{(2)}$ satisfies the $P(z_0)$-compatibility condition even if $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ is not assumed to satisfy Part (b) of the $P(z_0)$-local grading restriction condition. Finally, we comment that the proof above certainly also proves (14.51) in [H] as a special case.

Remark 9.19 We now use the part of the proof of Theorem 9.17 from (9.46) to (9.51) to prove the part of Proposition 9.8 on the uniqueness of the elements $\lambda_n^{(2)}$, $n \in \mathbb{R}$, with the properties indicated in Part (a) of the $P^{(2)}(z)$-local grading restriction condition; the other three cases are handled the same way. Using the proof from (9.46) to (9.51) with z_0 and $-l^0(z)$ replaced by z and z', respectively, we have that the sum of

$$\sum_{n \in \mathbb{R}} (e^{z'L'_{P(z)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} e^{nz'} \left(\left(\sum_{i=0}^{N} \frac{(z')^i}{i!} (L'_{P(z)}(0) - L'_{P(z)}(0)_s)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right) \qquad (9.105)$$

is an analytic function of z' for z' in an open neighborhood of 0, that its k-th derivative with respect to z' at $z' = 0$ is the sum of the absolutely convergent series

$$\sum_{n \in \mathbb{R}} (L'_{P(z)}(0)^k \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)}), \qquad (9.106)$$

and that the iterated sum on the right-hand side of (9.105) equals the corresponding double sum, absolutely convergent in a suitably small neighborhood of $z' = 0$ independent of $w_{(1)}$ and $w_{(2)}$. Using (5.110) repeatedly and then using (iii) of Part (a) of the $P^{(2)}(z)$-local grading restriction condition, we see that (9.106) is equal to

$$\sum_{i=0}^{k} \binom{k}{i} \sum_{n \in \mathbb{R}} \lambda_n^{(2)} ((L(0) + zL(-1))^{k-i} w_{(1)} \otimes L(0)^i w_{(2)})$$

$$= \sum_{i=0}^{k} \binom{k}{i} \mu_{\lambda, w_{(3)}}^{(2)} ((L(0) + zL(-1))^{k-i} w_{(1)} \otimes L(0)^i w_{(2)}). \qquad (9.107)$$

Since the right-hand side of (9.107) is independent of $\lambda_n^{(2)}$, $n \in \mathbb{R}$, the analytic function obtained from the double sum corresponding to (9.105) is also independent of $\lambda_n^{(2)}$, $n \in \mathbb{R}$, that is, if the formal series $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ and $\sum_{n \in \mathbb{R}} \tilde{\lambda}_n^{(2)}$ both satisfy Part (a) of the $P^{(2)}(z)$-local grading restriction condition, then

$$\sum_{n \in \mathbb{R}} (e^{z'L'_{P(z)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

and

$$\sum_{n\in\mathbb{R}}(e^{z'L'_{P(z)}(0)}\tilde{\lambda}_n^{(2)})(w_{(1)}\otimes w_{(2)})$$

are analytic functions equal to each other in a suitably small open neighborhood of $z'=0$. Thus in this neighborhood,

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{N}e^{nz'}\frac{(z')^i}{i!}((L'_{P(z)}(0)-L'_{P(z)}(0)_s)^i(\lambda_n^{(2)}-\tilde{\lambda}_n^{(2)}))(w_{(1)}\otimes w_{(2)})$$

$$=\sum_{n\in\mathbb{R}}(e^{z'L'_{P(z)}(0)}(\lambda_n^{(2)}-\tilde{\lambda}_n^{(2)}))(w_{(2)}\otimes w_{(3)})$$

$$=0,\tag{9.108}$$

and so by Proposition 7.8, $\mathbb{R}\times\{0,\ldots,N\}$ being a unique expansion set, we have

$$((L'_{P(z)}(0)-L'_{P(z)}(0)_s)^i(\lambda_n^{(2)}-\tilde{\lambda}_n^{(2)}))(w_{(1)}\otimes w_{(2)})=0$$

for $n\in\mathbb{R}$ and $i=0,\ldots,N$. In particular (for $i=0$),

$$\lambda_n^{(2)}-\tilde{\lambda}_n^{(2)}=0$$

for $n\in\mathbb{R}$, proving the uniqueness.

We will be invoking the uniqueness (Proposition 9.8) and the bilinearity (Corollary 9.9) of the elements $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ below.

We now relate Proposition 9.13 and Theorem 9.17 to $\boxtimes_{P(z)}$ and $\boxtimes_{P(z)}$ for suitable $z\in\mathbb{C}^\times$; recall Definitions 5.31 and 4.15. We will sometimes use Definition 9.14 and Remark 9.15. First we relate Proposition 9.13 to $\boxtimes_{P(z)}$, and this will serve as motivation for Corollary 9.21, in which we relate Theorem 9.17 to $\boxtimes_{P(z)}$.

Remark 9.20 Assume that $\mathcal{C}$ is closed under images. In the setting and under all the assumptions of Proposition 9.13, we have (according to this result): If $\lambda=(I_1\circ(1_{W_1}\otimes I_2))'(w'_{(4)})$ (respectively, $\lambda=(I^1\circ(I^2\otimes 1_{W_3}))'(w'_{(4)})$), then $W^{(1)}_{\lambda,w_{(1)}}$ (respectively, $W^{(2)}_{\lambda,w_{(3)}}$) is a generalized V-submodule of an object of $\mathcal{C}$ included in $(W_2\otimes W_3)^*$ (respectively, included in $(W_1\otimes W_2)^*$), and in particular, for each $n\in\mathbb{R}$ the $P(z_2)$-generalized weight vector $\lambda_n^{(1)}$ (respectively, the $P(z_0)$-generalized weight vector $\lambda_n^{(2)}$), which generates a generalized V-submodule of $W^{(1)}_{\lambda,w_{(1)}}$ (respectively, of $W^{(2)}_{\lambda,w_{(3)}}$), also generates a generalized V-submodule of an object of $\mathcal{C}$ included in $(W_2\otimes W_3)^*$ (respectively, $(W_1\otimes W_2)^*$). Hence by Proposition 5.36,

$$\lambda_n^{(1)}\in W_2\boxtimes_{P(z_2)}W_3$$

and

$$\lambda_n^{(2)}\in W_1\boxtimes_{P(z_0)}W_2$$

for each $n\in\mathbb{R}$.

Invoking the last assertion of Proposition 5.36, we have the following corollary of Theorem 9.17:

Corollary 9.21 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that*

$$|z_1| > |z_2| > |z_0| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$ and let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Let $w'_{(4)} \in W'_4$.

1. *Suppose that*

$$\lambda = (I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$$

satisfies the (full) $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For any $w_{(3)} \in W_3$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ be the (unique) series weakly absolutely convergent to

$$\mu_{\lambda, w_{(3)}}^{(2)} \in (W_1 \otimes W_2)^*$$

as indicated in the $P^{(2)}(z_0)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition). If for each $n \in \mathbb{R}$ the generalized V-submodule of the generalized V-module $W_{\lambda, w_{(3)}}^{(2)}$ (given by Theorem 9.17) generated by $\lambda_n^{(2)}$ is a generalized V-submodule of some object of $\mathcal{C}$ (depending on n) included in $(W_1 \otimes W_2)^*$, then

$$\lambda_n^{(2)} \in W_1 \boxtimes_{P(z_0)} W_2.$$

2. *Analogously, suppose that*

$$\lambda = (I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)})$$

satisfies the (full) $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For any $w_{(1)} \in W_1$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ be the (unique) series weakly absolutely convergent to

$$\mu_{\lambda, w_{(1)}}^{(1)} \in (W_2 \otimes W_3)^*$$

as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition). If for each $n \in \mathbb{R}$ the generalized V-submodule of the generalized V-module $W_{\lambda, w_{(1)}}^{(1)}$ (given by Theorem 9.17) generated by $\lambda_n^{(1)}$ is a generalized V-submodule of some object of $\mathcal{C}$ (depending on n) included in $(W_2 \otimes W_3)^*$, then

$$\lambda_n^{(1)} \in W_2 \boxtimes_{P(z_2)} W_3.$$

$\square$

Next we shall express a product of suitable intertwining maps as an iterate and vice versa. This is accomplished in Theorem 9.23 below. We shall actually carry this out only for the case of expressing a product as an iterate, which is Part 1 of Theorem 9.23; this case is based on Lemma 9.22 below. Expresing an iterate as a product (Part 2 of the theorem) is proved analogously. We start with hypotheses for the lemma and the theorem.

Assume that $\mathcal{C}$ is closed under images, that the convergence condition for intertwining maps in $\mathcal{C}$ holds, and that

$$|z_1| > |z_2| > |z_0| > 0.$$

Let W_1, W_2, W_3, W_4 and M_1 be objects of $\mathcal{C}$ and let I_1 and I_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively. Set

$$G = (I_1 \circ (1_{W_1} \otimes I_2))' \in \mathrm{Hom}(W_4', (W_1 \otimes W_2 \otimes W_3)^*)$$

(cf. Remark 9.11). Suppose that $W_1 \boxtimes_{P(z_0)} W_2$ exists in $\mathcal{C}$ and that for each $w_{(4)}' \in W_4'$, $G(w_{(4)}')$, as in Corollary 9.21, satisfies the $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w_{(3)} \in W_3$, let

$$\sum_{n \in \mathbb{R}} \lambda_n^{(2)}(w_{(4)}', w_{(3)}) \tag{9.109}$$

be the (unique) series weakly absolutely convergent to

$$\mu_{G(w_{(4)}'), w_{(3)}}^{(2)} \in (W_1 \otimes W_2)^* \tag{9.110}$$

as indicated in the $P^{(2)}(z_0)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition). Suppose further that for each $n \in \mathbb{R}$, $w_{(4)}' \in W_4'$ and $w_{(3)} \in W_3$, the generalized V-submodule of $W_{G(w_{(4)}'), w_{(3)}}^{(2)}$ generated by $\lambda_n^{(2)}(w_{(4)}', w_{(3)})$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$. Using Part 1 of Corollary 9.21, which is based on and follows from Part 1 of Theorem 9.17, we shall now prove that the product $I_1 \circ (1_{W_1} \otimes I_2)$ of the intertwining maps I_1 and I_2 can be written as an iterate of suitable intertwining maps, which is Part 1 of Theorem 9.23 below. First we formulate and prove a lemma under these assumptions. This lemma is the core of the proof of the theorem.

Recall from Proposition 5.37 that since $\mathcal{C}$ is closed under images, the existence of $W_1 \boxtimes_{P(z_0)} W_2$ in $\mathcal{C}$ implies that $W_1 \boxtimes_{P(z_0)} W_2$ is an object of $\mathcal{C}$ and that

$$W_1 \boxtimes_{P(z_0)} W_2 = (W_1 \boxtimes_{P(z_0)} W_2)'. \tag{9.111}$$

By Corollary 9.21, for any $w_{(4)}' \in W_4'$ and $w_{(3)} \in W_3$, $\lambda_n^{(2)}(w_{(4)}', w_{(3)})$ is an element (of generalized weight n) of

$$W_1 \boxtimes_{P(z_0)} W_2 \subset (W_1 \otimes W_2)^*$$

for $n \in \mathbb{R}$. Thus we have the element

$$\widetilde{\mu}_{G(w_{(4)}'), w_{(3)}}^{(2)} \in \overline{W_1 \boxtimes_{P(z_0)} W_2}, \tag{9.112}$$

the formal completion of the generalized V-module (and object of $\mathcal{C}$) $W_1 \boxtimes_{P(z_0)} W_2$, whose homogeneous components are the $P(z_0)$-generalized weight vectors

$$\lambda_n^{(2)}(w'_{(4)}, w_{(3)}) \in W_1 \boxtimes_{P(z_0)} W_2,$$

$n \in \mathbb{R}$. Recalling the notation

$$\pi_n : \overline{W_1 \boxtimes_{P(z_0)} W_2} \to W_1 \boxtimes_{P(z_0)} W_2$$

from Definition 2.18, we have

$$\lambda_n^{(2)}(w'_{(4)}, w_{(3)}) = \pi_n(\widetilde{\mu}_{G(w'_{(4)}),w_{(3)}}^{(2)}), \tag{9.113}$$

and so from (9.109) and (9.110) we have the weakly absolutely convergent series

$$\mu_{G(w'_{(4)}),w_{(3)}}^{(2)} = \sum_{n \in \mathbb{R}} \pi_n(\widetilde{\mu}_{G(w'_{(4)}),w_{(3)}}^{(2)}) = \sum_{n \in \mathbb{R}} \lambda_n^{(2)}(w'_{(4)}, w_{(3)}). \tag{9.114}$$

Note the distinction between the different sums (9.112) and (9.114) of the same elements $\lambda_n^{(2)}(w'_{(4)}, w_{(3)})$; they take place in different spaces. Recall from Definition 2.32 and (9.111) that we have a canonical pairing

$$\langle \cdot, \cdot \rangle_{W_1 \boxtimes_{P(z_0)} W_2}$$

between $W_1 \boxtimes_{P(z_0)} W_2$ and $\overline{W_1 \boxtimes_{P(z_0)} W_2}$. By Corollary 9.9, the element (9.112) depends bilinearly on $w'_{(4)}$ and $w_{(3)}$, so that we have a linear map

$$\widetilde{G} : W_1 \boxtimes_{P(z_0)} W_2 \to (W'_4 \otimes W_3)^*$$

determined by the condition

$$\widetilde{G}(w)(w'_{(4)} \otimes w_{(3)}) = \langle w, \widetilde{\mu}_{G(w'_{(4)}),w_{(3)}}^{(2)} \rangle_{W_1 \boxtimes_{P(z_0)} W_2} \tag{9.115}$$

for $w \in W_1 \boxtimes_{P(z_0)} W_2$. Moreover, generalizing the corresponding lemma in [H], we have the following lemma, in which $\tau_{Q(z_2)}$ (recall Section 5.1, in particular, Definition 5.51) appears naturally:

Lemma 9.22 *Under these assumptions, the linear map*

$$\widetilde{G} \in \operatorname{Hom}(W_1 \boxtimes_{P(z_0)} W_2, (W'_4 \otimes W_3)^*)$$

intertwines the actions $\tau_{W_1 \boxtimes_{P(z_0)} W_2}$ and $\tau_{Q(z_2)}$ of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_2 + t)^{-1}]$, and also the corresponding $\mathfrak{sl}(2)$ actions, on $W_1 \boxtimes_{P(z_0)} W_2$ and on $(W'_4 \otimes W_3)^*$ (recall Section 5.1 for these actions).

Proof We shall prove the assertion about $\tau_{W_1 \boxtimes_{P(z_0)} W_2}$ and $\tau_{Q(z_2)}$, and at the end of this proof we shall briefly comment that one can similarly prove the assertion about $\mathfrak{sl}(2)$ by considering appropriate aspects of the case $v = \omega$.

As in Proposition 5.37, we shall denote the vertex operator map for $W_1 \boxtimes_{P(z_0)} W_2$ by $Y_{P(z_0)}$. By (9.115) and the definition (5.154) of the $\tau_{Q(z_2)}$-action (see also (5.156)), we need to show that

$$z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) \widetilde{G}(Y_{P(z_0)}(v, x_0)w)$$
$$= \tau_{Q(z_2)} \left(z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) Y_t(v, x_0) \right) \widetilde{G}(w) \qquad (9.116)$$

for $v \in V$ and $w \in W_1 \boxtimes_{P(z_0)} W_2$, or equivalently, that

$$\left\langle z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) Y_{P(z_0)}(v, x_0)w, \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}} \right\rangle_{W_1 \boxtimes_{P(z_0)} W_2}$$
$$= \left(\tau_{Q(z_2)} \left(z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) Y_t(v, x_0) \right) \widetilde{G}(w) \right) (w'_{(4)} \otimes w_{(3)}) \qquad (9.117)$$

for $v \in V$, $w \in W_1 \boxtimes_{P(z_0)} W_2$, $w'_{(4)} \in W'_4$, $w_{(3)} \in W_3$. Note that the left-hand sides of (9.116) and of (9.117) involve only finitely many negative powers of x_0.

By Proposition 4.23, we need only prove our assertion for

$$w = \pi_n (w_{(1)} \boxtimes_{P(z_0)} w_{(2)})$$

for any $n \in \mathbb{R}$ and $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. (Again recall the notation π_n from Definition 2.18.)

By (5.139), for $n \in \mathbb{R}$ we have

$$(\lambda^{(2)}_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)})$$
$$= \langle \lambda^{(2)}_n(w'_{(4)}, w_{(3)}), w_{(1)} \boxtimes_{P(z_0)} w_{(2)} \rangle_{W_1 \boxtimes_{P(z_0)} W_2}$$
$$= \langle \lambda^{(2)}_n(w'_{(4)}, w_{(3)}), \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \rangle$$
$$= \langle \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}, \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \rangle, \qquad (9.118)$$

where the last pairing is between $\overline{W_1 \boxtimes_{P(z_0)} W_2}$ and $W_1 \boxtimes_{P(z_0)} W_2$.

Recalling (5.24), (2.57), (2.73) and the definition (5.85) of $Y'_{P(z_0)}$ (see also (5.87)), we define

$$Y'^o_{P(z_0)}(v, x_0) : (W_1 \otimes W_2)^* \to (W_1 \otimes W_2)^*[[x_0, x_0^{-1}]]$$

by

$$Y'^o_{P(z_0)}(v, x_0)\mu = \tau_{P(z_0)}(Y_t^o(v, x_0))\mu$$
$$= Y'_{P(z_0)}(e^{x_0 L(1)}(-x_0^{-2})^{L(0)} v, x_0^{-1})\mu \qquad (9.119)$$

for $\mu \in (W_1 \otimes W_2)^*$. Then

$$Y'^o_{P(z_0)}(v, x_0)\Big|_{W_1 \boxtimes_{P(z_0)} W_2} = Y^o_{W_1 \boxtimes_{P(z_0)} W_2}(v, x_0). \tag{9.120}$$

We have natural maps

$$Y'_{P(z_0)}(v, x_0) : \overline{W_1 \boxtimes_{P(z_0)} W_2} \to \overline{W_1 \boxtimes_{P(z_0)} W_2}[[x_0, x_0^{-1}]]$$

and

$$Y'^o_{P(z_0)}(v, x_0) : \overline{W_1 \boxtimes_{P(z_0)} W_2} \to \overline{W_1 \boxtimes_{P(z_0)} W_2}[[x_0, x_0^{-1}]];$$

and

$$\pi_n(Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)} \otimes w_{(2)})$$

$$= \langle \pi_n(Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})), w_{(1)} \boxtimes_{P(z_0)} w_{(2)} \rangle$$

$$= \langle Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}}, \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \rangle \tag{9.121}$$

for $n \in \mathbb{R}$ (cf. (9.118)).

Now taking

$$w = \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)})$$

in the left-hand side of (9.117) and using the definition of $Y_{P(z_0)}$ (see Proposition 5.37), (9.121) and (9.120) (and, as we have done above, dropping the subscripts for the pairings), we see that the left-hand side of (9.117) becomes

$$\left\langle z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) Y_{P(z_0)}(v, x_0)(\pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)})), \widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}} \right\rangle$$

$$= z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) \langle Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}}, \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \rangle$$

$$= z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n(Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)} \otimes w_{(2)}).$$

$$\tag{9.122}$$

Recall that this expression involves only finitely many negative powers of x_0.

Taking

$$w = \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)})$$

in the right-hand side of (9.117) and using the definition (5.156) of $\tau_{Q(z_2)}$, (9.115) and (9.118), we obtain

$$\left(\tau_{Q(z_2)}\left(z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) Y_t(v, x_0)\right)\widetilde{G}(\pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}))\right)(w'_{(4)} \otimes w_{(3)})$$

$$= x_0^{-1}\delta\left(\frac{x_1 - z_2}{x_0}\right)(\widetilde{G}(\pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)})))(Y'^o_4(v, x_1)w'_{(4)} \otimes w_{(3)})$$

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$$-x_0^{-1}\delta\Big(\frac{z_2-x_1}{-x_0}\Big)(\widetilde{G}(\pi_n(w_{(1)}\boxtimes_{P(z_0)}w_{(2)})))(w_{(4)}'\otimes Y_3(v,x_1)w_{(3)})$$

$$=x_0^{-1}\delta\Big(\frac{x_1-z_2}{x_0}\Big)\langle\pi_n(w_{(1)}\boxtimes_{P(z_0)}w_{(2)}),\widetilde{\mu}^{(2)}_{G(Y_4'^o(v,x_1)w_{(4)}'),w_{(3)}}\rangle$$

$$-x_0^{-1}\delta\Big(\frac{z_2-x_1}{-x_0}\Big)\langle\pi_n(w_{(1)}\boxtimes_{P(z_0)}w_{(2)}),\widetilde{\mu}^{(2)}_{G(w_{(4)}'),Y_3(v,x_1)w_{(3)}}\rangle$$

$$=x_0^{-1}\delta\Big(\frac{x_1-z_2}{x_0}\Big)(\lambda_n^{(2)}(Y_4'^o(v,x_1)w_{(4)}',w_{(3)}))(w_{(1)}\otimes w_{(2)})$$

$$-x_0^{-1}\delta\Big(\frac{z_2-x_1}{-x_0}\Big)(\lambda_n^{(2)}(w_{(4)}',Y_3(v,x_1)w_{(3)}))(w_{(1)}\otimes w_{(2)}). \qquad (9.123)$$

In order to prove the equality of (9.122) and (9.123), we will consider the sums of both expressions over $n\in\mathbb{R}$.

By the definitions (9.119) and (5.87) of $Y_{P(z_0)}'^o$ and $Y_{P(z_0)}'$, we have, using (5.25),

$$(Y_{P(z_0)}'^o(v,x_0)\mu)(w_{(1)}\otimes w_{(2)})$$
$$=(Y_{P(z_0)}'(e^{x_0L(1)}(-x_0^{-2})^{L(0)}v,x_0^{-1})\mu)(w_{(1)}\otimes w_{(2)})$$
$$=\mu(w_{(1)}\otimes Y_2(v,x_0)w_{(2)})$$
$$+\operatorname{Res}_{x_2}z_0^{-1}\delta\Big(\frac{x_0-x_2}{z_0}\Big)\mu(Y_1(v,x_2)w_{(1)}\otimes w_{(2)}) \qquad (9.124)$$

for $\mu\in(W_1\otimes W_2)^*$, $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$. Taking

$$\mu=\mu^{(2)}_{G(w_{(4)}'),w_{(3)}},$$

we thus have

$$(Y_{P(z_0)}'^o(v,x_0)\mu^{(2)}_{G(w_{(4)}'),w_{(3)}})(w_{(1)}\otimes w_{(2)})$$
$$=\mu^{(2)}_{G(w_{(4)}'),w_{(3)}}(w_{(1)}\otimes Y_2(v,x_0)w_{(2)})$$
$$+\operatorname{Res}_{x_2}z_0^{-1}\delta\Big(\frac{x_0-x_2}{z_0}\Big)\mu^{(2)}_{G(w_{(4)}'),w_{(3)}}(Y_1(v,x_2)w_{(1)}\otimes w_{(2)}).$$

From (9.114), whose right-hand side is weakly absolutely convergent,

$$(Y_{P(z_0)}'^o(v,x_0)\sum_{n\in\mathbb{R}}\pi_n(\widetilde{\mu}^{(2)}_{G(w_{(4)}'),w_{(3)}}))(w_{(1)}\otimes w_{(2)})$$
$$=\sum_{n\in\mathbb{R}}(\pi_n(\widetilde{\mu}^{(2)}_{G(w_{(4)}'),w_{(3)}})(w_{(1)}\otimes Y_2(v,x_0)w_{(2)}))$$
$$+\operatorname{Res}_{x_2}z_0^{-1}\delta\Big(\frac{x_0-x_2}{z_0}\Big)\sum_{n\in\mathbb{R}}(\pi_n(\widetilde{\mu}^{(2)}_{G(w_{(4)}'),w_{(3)}})(Y_1(v,x_2)w_{(1)}\otimes w_{(2)}))$$
$$=\sum_{n\in\mathbb{R}}(\pi_n(\widetilde{\mu}^{(2)}_{G(w_{(4)}'),w_{(3)}})(w_{(1)}\otimes Y_2(v,x_0)w_{(2)}))$$
$$+\sum_{n\in\mathbb{R}}\operatorname{Res}_{x_2}z_0^{-1}\delta\Big(\frac{x_0-x_2}{z_0}\Big)(\pi_n(\widetilde{\mu}^{(2)}_{G(w_{(4)}'),w_{(3)}})(Y_1(v,x_2)w_{(1)}\otimes w_{(2)})),$$

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since in the second term, the coefficient of each monomial in x_2 involves only the single infinite sum $\sum_{n \in \mathbb{R}}$, and so we can switch $z_0^{-1}\delta\left(\frac{x_0-x_2}{z_0}\right)$ and $\sum_{n \in \mathbb{R}}$. Thus

$$
\begin{aligned}
&(Y_{P(z_0)}^{\prime o}(v, x_0) \sum_{n \in \mathbb{R}} \pi_n(\widetilde{\mu}_{G(w'_{(4)}),w_{(3)}}^{(2)}))(w_{(1)} \otimes w_{(2)}) \\
&= \sum_{n \in \mathbb{R}} (Y_{P(z_0)}^{\prime o}(v, x_0) \pi_n(\widetilde{\mu}_{G(w'_{(4)}),w_{(3)}}^{(2)}))(w_{(1)} \otimes w_{(2)}),
\end{aligned}
\tag{9.125}
$$

with absolute convergence for the coefficient of each monomial in x_0.

Now the product of the first term in the right-hand side of (9.124) with $z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)$ exists algebraically, and since the product

$$
z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) z_0^{-1}\delta\left(\frac{x_0 - x_2}{z_0}\right)
\tag{9.126}
$$

exists in the sense of absolute convergence, by (8.5) in Proposition 8.1 (since $|z_2| > |z_0| > 0$), the product of the second term in the right-hand side of (9.124) with $z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)$ exists in the sense of absolute convergence. (That is, the sum over the integral powers of x_0 obtained from extracting the coefficient of any monomial in x_0, x_1 and x_2 is absolutely convergent.) Thus the product of the left-hand side of (9.124) with $z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)$ also exists in the sense of absolute convergence. Again taking

$$
\mu = \mu_{G(w'_{(4)}),w_{(3)}}^{(2)},
$$

we thus have

$$
\begin{aligned}
&z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right)(Y_{P(z_0)}^{\prime o}(v, x_0)\mu_{G(w'_{(4)}),w_{(3)}}^{(2)})(w_{(1)} \otimes w_{(2)}) \\
&= z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right)\mu_{G(w'_{(4)}),w_{(3)}}^{(2)}(w_{(1)} \otimes Y_2(v, x_0)w_{(2)}) \\
&\quad + z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right)\operatorname{Res}_{x_2} z_0^{-1}\delta\left(\frac{x_0 - x_2}{z_0}\right)\mu_{G(w'_{(4)}),w_{(3)}}^{(2)}(Y_1(v, x_2)w_{(1)} \otimes w_{(2)}),
\end{aligned}
\tag{9.127}
$$

in the sense of absolute convergence. We will need a variant of this formula, with the left-hand side replaced by $\sum_{n \in \mathbb{R}}$ applied to (9.122) (see (9.131) below).

From (9.114), the right-hand side of (9.127) is equal to

$$
\begin{aligned}
&z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right)\sum_{n \in \mathbb{R}} \pi_n(\widetilde{\mu}_{G(w'_{(4)}),w_{(3)}}^{(2)})(w_{(1)} \otimes Y_2(v, x_0)w_{(2)}) \\
&\quad + z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right)\operatorname{Res}_{x_2} z_0^{-1}\delta\left(\frac{x_0 - x_2}{z_0}\right)\sum_{n \in \mathbb{R}} \pi_n(\widetilde{\mu}_{G(w'_{(4)}),w_{(3)}}^{(2)})(Y_1(v, x_2)w_{(1)} \otimes w_{(2)})
\end{aligned}
$$

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$$= z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\sum_{n\in\mathbb{R}}\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)}\otimes Y_2(v,x_0)w_{(2)})$$

$$+\operatorname{Res}_{x_2}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)z_0^{-1}\delta\left(\frac{x_0-x_2}{z_0}\right)\sum_{n\in\mathbb{R}}\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(Y_1(v,x_2)w_{(1)}\otimes w_{(2)}),$$
(9.128)

where the sums over $n\in\mathbb{R}$ are absolutely convergent.

In the first term in the right-hand side of (9.128), the coefficient of each monomial in x_0 and x_1 involves only the single infinite sum $\sum_{n\in\mathbb{R}}$, and so we can switch $z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)$ and $\sum_{n\in\mathbb{R}}$ in this term. In the second term, both (9.126) and

$$\sum_{n\in\mathbb{R}}\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(Y_1(v,x_2)w_{(1)}\otimes w_{(2)})$$

are formal series in x_0, x_1 and x_2 each of whose coefficients is an absolutely convergent series, and both series are truncated from below in powers of x_2. Thus the double sums obtained from the coefficients of the product of these two formal series in x_0, x_1 and x_2 are also absolutely convergent and in particular, we can switch (9.126) and $\sum_{n\in\mathbb{R}}$ in the second term. So we see, using (9.124), that the right-hand side of (9.128) is equal to

$$\sum_{n\in\mathbb{R}}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)}\otimes Y_2(v,x_0)w_{(2)})$$

$$+\sum_{n\in\mathbb{R}}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\operatorname{Res}_{x_2}z_0^{-1}\delta\left(\frac{x_0-x_2}{z_0}\right)\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(Y_1(v,x_2)w_{(1)}\otimes w_{(2)})$$

$$=\sum_{n\in\mathbb{R}}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)(Y'^o_{P(z_0)}(v,x_0)\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}}))(w_{(1)}\otimes w_{(2)}),\qquad(9.129)$$

and the corresponding double sums in the two terms in the left-hand side and thus the corresponding double sums in the right-hand side are all absolutely convergent. Hence, using the fact that the coefficient of each monomial in the right-hand side of (9.125) is absolutely convergent, we obtain that the right-hand side of (9.129) is equal to

$$\sum_{n\in\mathbb{R}}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)(Y'^o_{P(z_0)}(v,x_0)\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}}))(w_{(1)}\otimes w_{(2)})$$

$$=z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\sum_{n\in\mathbb{R}}(Y'^o_{P(z_0)}(v,x_0)\pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}}))(w_{(1)}\otimes w_{(2)})$$

$$=z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\sum_{n\in\mathbb{R}}\pi_n(Y'^o_{P(z_0)}(v,x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)}\otimes w_{(2)})$$

$$=\sum_{n\in\mathbb{R}}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\pi_n(Y'^o_{P(z_0)}(v,x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)}\otimes w_{(2)}),\qquad(9.130)$$

and we continue to have multiple absolute convergence. Note that on the right-hand side, for each $n \in \mathbb{R}$ the sum over the integral powers of x_0 is a finite sum, in view of our comment after (9.122).

By the results from (9.128) to (9.130) we obtain

$$\sum_{n \in \mathbb{R}} z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n(Y_{P(z_0)}^{\prime o}(v, x_0) \widetilde{\mu}_{G(w'_{(4)}), w_{(3)}}^{(2)})(w_{(1)} \otimes w_{(2)})$$

$$= z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \mu_{G(w'_{(4)}), w_{(3)}}^{(2)}(w_{(1)} \otimes Y_2(v, x_0) w_{(2)})$$

$$+ z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \operatorname{Res}_{x_2} z_0^{-1} \delta\left(\frac{x_0 - x_2}{z_0}\right) \mu_{G(w'_{(4)}), w_{(3)}}^{(2)}(Y_1(v, x_2) w_{(1)} \otimes w_{(2)})$$

$$\tag{9.131}$$

(with absolute convergence). Note that in this variant of (9.127), the left-hand side is the sum over $n \in \mathbb{R}$ of the right-hand side of (9.122).)

We now relate the right-hand side of (9.131) to (9.123). By the definitions of $\mu_{G(Y_4^{\prime o}(v, x_1) w'_{(4)}), w_{(3)}}^{(2)}$ and $\mu_{G(w'_{(4)}), Y_3(v, x_1) w_{(3)}}^{(2)}$ (cf. (9.110); recall (2.57) and (2.73)), we have

$$x_0^{-1} \delta\left(\frac{x_1 - z_2}{x_0}\right) \mu_{G(Y_4^{\prime o}(v, x_1) w'_{(4)}), w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)})$$

$$- x_0^{-1} \delta\left(\frac{z_2 - x_1}{-x_0}\right) \mu_{G(w'_{(4)}), Y_3(v, x_1) w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)})$$

$$= x_0^{-1} \delta\left(\frac{x_1 - z_2}{x_0}\right) \langle Y_4^{\prime o}(v, x_1) w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle$$

$$- x_0^{-1} \delta\left(\frac{z_2 - x_1}{-x_0}\right) \langle w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_1) w_{(3)}) \rangle$$

$$= x_0^{-1} \delta\left(\frac{x_1 - z_2}{x_0}\right) \langle w'_{(4)}, Y_4(v, x_1)(I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle$$

$$- x_0^{-1} \delta\left(\frac{z_2 - x_1}{-x_0}\right) \langle w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_1) w_{(3)}) \rangle. \tag{9.132}$$

Now using the formula obtained by taking Res_{x_1} of (8.9) then replacing x_0 by x_1 and x_2 by x_0, we see that the right-hand side of (9.132) is equal to

$$z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \langle w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes Y_2(v, x_0) w_{(2)} \otimes w_{(3)}) \rangle$$

$$+ x_0^{-1} \delta\left(\frac{x_1 - z_2}{x_0}\right) \operatorname{Res}_{x_2} z_1^{-1} \delta\left(\frac{x_1 - x_2}{z_1}\right) \langle w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))(Y_1(v, x_2) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle$$

$$= z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \mu_{G(w'_{(4)}), w_{(3)}}^{(2)}(w_{(1)} \otimes Y_2(v, x_0) w_{(2)})$$

$$+ x_0^{-1} \delta\left(\frac{x_1 - z_2}{x_0}\right) \operatorname{Res}_{x_2} z_1^{-1} \delta\left(\frac{x_1 - x_2}{z_1}\right) \mu_{G(w'_{(4)}), w_{(3)}}^{(2)}(Y_1(v, x_2) w_{(1)} \otimes w_{(2)}). \tag{9.133}$$

Using (8.5) and (8.4), we obtain

$$z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)z_0^{-1}\delta\left(\frac{x_0-x_2}{z_0}\right) = x_1^{-1}\delta\left(\frac{z_1+x_2}{x_1}\right)x_0^{-1}\delta\left(\frac{z_0+x_2}{x_0}\right)$$

$$= z_1^{-1}\delta\left(\frac{x_1-x_2}{z_1}\right)x_0^{-1}\delta\left(\frac{x_1-z_2}{x_0}\right),$$

so that the right-hand side of (9.133) is equal to

$$z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\mu^{(2)}_{G(w'_{(4)}),w_{(3)}}(w_{(1)}\otimes Y_2(v,x_0)w_{(2)})$$

$$+z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\operatorname{Res}_{x_2}z_0^{-1}\delta\left(\frac{x_0-x_2}{z_0}\right)\mu^{(2)}_{G(w'_{(4)}),w_{(3)}}(Y_1(v,x_2)w_{(1)}\otimes w_{(2)})$$

$$\tag{9.134}$$

From (9.132), (9.133) and (9.134), we obtain that the right-hand side of (9.131) equals

$$x_0^{-1}\delta\left(\frac{x_1-z_2}{x_0}\right)\mu^{(2)}_{G(Y_4'^o(v,x_1)w'_{(4)}),w_{(3)}}(w_{(1)}\otimes w_{(2)})$$

$$-x_0^{-1}\delta\left(\frac{z_2-x_1}{-x_0}\right)\mu^{(2)}_{G(w'_{(4)}),Y_3(v,x_1)w_{(3)}}(w_{(1)}\otimes w_{(2)}),$$

so that by (9.131),

$$\sum_{n\in\mathbb{R}}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\pi_n(Y'^o_{P(z_0)}(v,x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)}\otimes w_{(2)})$$

$$= x_0^{-1}\delta\left(\frac{x_1-z_2}{x_0}\right)\mu^{(2)}_{G(Y_4'^o(v,x_1)w'_{(4)}),w_{(3)}}(w_{(1)}\otimes w_{(2)})$$

$$-x_0^{-1}\delta\left(\frac{z_2-x_1}{-x_0}\right)\mu^{(2)}_{G(w'_{(4)}),Y_3(v,x_1)w_{(3)}}(w_{(1)}\otimes w_{(2)})\tag{9.135}$$

for all $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$. The right-hand side of (9.135) is equal to

$$x_0^{-1}\delta\left(\frac{x_1-z_2}{x_0}\right)\sum_{n\in\mathbb{R}}\lambda^{(2)}_n(Y_4'^o(v,x_1)w'_{(4)},w_{(3)})(w_{(1)}\otimes w_{(2)})$$

$$-x_0^{-1}\delta\left(\frac{z_2-x_1}{-x_0}\right)\sum_{n\in\mathbb{R}}\lambda^{(2)}_n(w'_{(4)},Y_3(v,x_1)w_{(3)})(w_{(1)}\otimes w_{(2)})\tag{9.136}$$

(recall (9.114)). Since the only infinite sums in (9.136) are those over $n\in\mathbb{R}$, we can move $\sum_{n\in\mathbb{R}}$ to the left to obtain from (9.135)

$$\sum_{n\in\mathbb{R}}z_2^{-1}\delta\left(\frac{x_1-x_0}{z_2}\right)\pi_n(Y'^o_{P(z_0)}(v,x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}})(w_{(1)}\otimes w_{(2)})$$

$$= \sum_{n\in\mathbb{R}}x_0^{-1}\delta\left(\frac{x_1-z_2}{x_0}\right)\lambda^{(2)}_n(Y_4'^o(v,x_1)w'_{(4)},w_{(3)})(w_{(1)}\otimes w_{(2)})$$

$$-\sum_{n\in\mathbb{R}}x_0^{-1}\delta\left(\frac{z_2-x_1}{-x_0}\right)\lambda^{(2)}_n(w'_{(4)},Y_3(v,x_1)w_{(3)})(w_{(1)}\otimes w_{(2)})\tag{9.137}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. That is, the sums over $n \in \mathbb{R}$ of (9.122) and (9.123) are equal.

We now set up an application of Proposition 7.8, by first establishing from (9.137) an equality of formal power series in y (see (9.143) below) and then specializing y to z' and proving and using certain convergence assertions. For $k \in \mathbb{N}$,

$$
\sum_{n \in \mathbb{R}} x_0^{-1} \delta\Big(\frac{x_1 - z_2}{x_0}\Big) \cdot
$$
$$
\cdot \sum_{i=0}^{k} \binom{k}{i} \lambda_n^{(2)}(Y_4'^o(v, x_1)w_{(4)}', w_{(3)})((L(0) + z_0 L(-1))^{k-i} w_{(1)} \otimes L(0)^i w_{(2)})
$$
$$
- \sum_{n \in \mathbb{R}} x_0^{-1} \delta\Big(\frac{z_2 - x_1}{-x_0}\Big) \cdot
$$
$$
\cdot \sum_{i=0}^{k} \binom{k}{i} \lambda_n^{(2)}(w_{(4)}', Y_3(v, x_1)w_{(3)})((L(0) + z_0 L(-1))^{k-i} w_{(1)} \otimes L(0)^i w_{(2)})
$$
$$
= \sum_{n \in \mathbb{R}} z_2^{-1} \delta\Big(\frac{x_1 - x_0}{z_2}\Big) \cdot
$$
$$
\cdot \sum_{i=0}^{k} \binom{k}{i} \pi_n(Y_{P(z_0)}'^o(v, x_0)\widetilde{\mu}_{G(w_{(4)}'), w_{(3)}}^{(2)})((L(0) + z_0 L(-1))^{k-i} w_{(1)} \otimes L(0)^i w_{(2)}),
$$

that is (by (5.110)),

$$
\sum_{n \in \mathbb{R}} x_0^{-1} \delta\Big(\frac{x_1 - z_2}{x_0}\Big) L_{P(z_0)}'(0)^k (\lambda_n^{(2)}(Y_4'^o(v, x_1)w_{(4)}', w_{(3)}))(w_{(1)} \otimes w_{(2)})
$$
$$
- \sum_{n \in \mathbb{R}} x_0^{-1} \delta\Big(\frac{z_2 - x_1}{-x_0}\Big) L_{P(z_0)}'(0)^k (\lambda_n^{(2)}(w_{(4)}', Y_3(v, x_1)w_{(3)}))(w_{(1)} \otimes w_{(2)})
$$
$$
= \sum_{n \in \mathbb{R}} z_2^{-1} \delta\Big(\frac{x_1 - x_0}{z_2}\Big) L_{P(z_0)}'(0)^k (\pi_n(Y_{P(z_0)}'^o(v, x_0)\widetilde{\mu}_{G(w_{(4)}'), w_{(3)}}^{(2)}))(w_{(1)} \otimes w_{(2)}),
$$

which gives

$$
\sum_{n \in \mathbb{R}} x_0^{-1} \delta\Big(\frac{x_1 - z_2}{x_0}\Big) e^{y L_{P(z_0)}'(0)} (\lambda_n^{(2)}(Y_4'^o(v, x_1)w_{(4)}', w_{(3)}))((w_{(1)} \otimes w_{(2)})
$$
$$
- \sum_{n \in \mathbb{R}} x_0^{-1} \delta\Big(\frac{z_2 - x_1}{-x_0}\Big) e^{y L_{P(z_0)}'(0)} (\lambda_n^{(2)}(w_{(4)}', Y_3(v, x_1)w_{(3)}))(w_{(1)} \otimes w_{(2)})
$$
$$
= \sum_{n \in \mathbb{R}} z_2^{-1} \delta\Big(\frac{x_1 - x_0}{z_2}\Big) e^{y L_{P(z_0)}'(0)} (\pi_n(Y_{P(z_0)}'^o(v, x_0)\widetilde{\mu}_{G(w_{(4)}'), w_{(3)}}^{(2)}))(w_{(1)} \otimes w_{(2)}).
$$
$$
(9.138)
$$

Now both sides of (9.138) are formal Laurent series in x_0 and x_1 with coefficients in $\mathbb{C}[[y]]$, and on the left-hand side, the coefficient of each monomial in x_0 and x_1 involves only finitely

many pairs of vectors in W_4' and W_3. Also, since $W_1 \boxtimes_{P(z_0)} W_2$ is an object of $\mathcal{C}$, by (9.112) and Assumption 7.11 there exists $K \in \mathbb{N}$ such that

$$(L'_{P(z_0)}(0) - n)^{K+1}(\pi_n(Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}})) = 0 \tag{9.139}$$

for $n \in \mathbb{R}$. Thus by Remark 9.7 and (9.139), for each pair $p, q \in \mathbb{Z}$ there exists $N_{p,q} \in \mathbb{N}$ with

$$N_{p,q} \geq K \tag{9.140}$$

such that

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n\in\mathbb{R}} x_0^{-1}\delta\Big(\frac{x_1 - z_2}{x_0}\Big) e^{yL'_{P(z_0)}(0)}(\lambda_n^{(2)}(Y'^o_4(v, x_1)w'_{(4)}, w_{(3)}))((w_{(1)} \otimes w_{(2)})$$

$$-\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n\in\mathbb{R}} x_0^{-1}\delta\Big(\frac{z_2 - x_1}{-x_0}\Big) e^{yL'_{P(z_0)}(0)}(\lambda_n^{(2)}(w'_{(4)}, Y_3(v, x_1)w_{(3)}))(w_{(1)} \otimes w_{(2)})$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n\in\mathbb{R}} x_0^{-1}\delta\Big(\frac{x_1 - z_2}{x_0}\Big) e^{ny} \cdot$$

$$\cdot \sum_{i=0}^{N_{p,q}} \frac{y^i}{i!}((L'_{P(z_0)}(0) - n)^i(\lambda_n^{(2)}(Y'^o_4(v, x_1)w'_{(4)}, w_{(3)}))))(w_{(1)} \otimes w_{(2)})$$

$$-\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n\in\mathbb{R}} x_0^{-1}\delta\Big(\frac{z_2 - x_1}{-x_0}\Big) e^{ny} \cdot$$

$$\cdot \sum_{i=0}^{N_{p,q}} \frac{y^i}{i!}((L'_{P(z_0)}(0) - n)^i(\lambda_n^{(2)}(w'_{(4)}, Y_3(v, x_1)w_{(3)}))))(w_{(1)} \otimes w_{(2)}) \tag{9.141}$$

and

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n\in\mathbb{R}} z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) e^{yL'_{P(z_0)}(0)}(\pi_n(Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}))(w_{(1)} \otimes w_{(2)})$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n\in\mathbb{R}} z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) e^{ny} \cdot$$

$$\cdot \sum_{i=0}^{N_{p,q}} \frac{y^i}{i!}((L'_{P(z_0)}(0) - n)^i(\pi_n(Y'^o_{P(z_0)}(v, x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}))))(w_{(1)} \otimes w_{(2)}). \tag{9.142}$$

In particular, we obtain from (9.138), (9.141) and (9.142)

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n\in\mathbb{R}} x_0^{-1}\delta\Big(\frac{x_1 - z_2}{x_0}\Big) e^{ny}.$$

$$\cdot \sum_{i=0}^{N_{p,q}} \frac{y^i}{i!}((L'_{P(z_0)}(0) - n)^i(\lambda_n^{(2)}(Y'^o_4(v, x_1)w'_{(4)}, w_{(3)}))))(w_{(1)} \otimes w_{(2)})$$

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$$-\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}x_0^p x_1^q \sum_{n\in\mathbb{R}} x_0^{-1}\delta\Big(\frac{z_2-x_1}{-x_0}\Big)e^{ny}\cdot$$

$$\cdot\sum_{i=0}^{N_{p,q}}\frac{y^i}{i!}((L'_{P(z_0)}(0)-n)^i(\lambda_n^{(2)}(w'_{(4)},Y_3(v,x_1)w_{(3)}))))(w_{(1)}\otimes w_{(2)})$$

$$=\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}x_0^p x_1^q \sum_{n\in\mathbb{R}} z_2^{-1}\delta\Big(\frac{x_1-x_0}{z_2}\Big)e^{ny}\cdot$$

$$\cdot\sum_{i=0}^{N_{p,q}}\frac{y^i}{i!}((L'_{P(z_0)}(0)-n)^i(\pi_n(Y^{\prime o}_{P(z_0)}(v,x_0)\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}}))))(w_{(1)}\otimes w_{(2)}). \quad (9.143)$$

We shall substitute z' for y in the two sides of (9.143), thus obtaining a common power series in z', and we shall show that this common power series is the power series expansion of two analytic functions, which must then be equal. We start with the left-hand side.

Using the part of the proof of Theorem 9.17 from (9.46) to (9.51) with $-l^0(z)$ replaced by z' (as we did earlier in Remark 9.19), we see using Proposition 7.9 that for each $i = 0,\ldots,N_{p,q}$, the series

$$\sum_{n\in\mathbb{R}}\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}x_0^p x_1^q x_0^{-1}\delta\Big(\frac{x_1-z_2}{x_0}\Big)e^{nz'}\cdot$$

$$\cdot((L'_{P(z_0)}(0)-n)^i(\lambda_n^{(2)}(Y^{\prime o}_4(v,x_1)w'_{(4)},w_{(3)}))))(w_{(1)}\otimes w_{(2)})$$

$$-\sum_{n\in\mathbb{R}}\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}x_0^p x_1^q x_0^{-1}\delta\Big(\frac{z_2-x_1}{-x_0}\Big)e^{nz'}\cdot$$

$$\cdot((L'_{P(z_0)}(0)-n)^i(\lambda_n^{(2)}(w'_{(4)},Y_3(v,x_1)w_{(3)}))))(w_{(1)}\otimes w_{(2)}) \quad (9.144)$$

is absolutely convergent in an open neighborhood of $z'=0$, and that by Lemma 7.7, (9.144) is in fact absolutely convergent to an analytic function of z' in this neighborhood.

The sum of (9.144) as an analytic function of z' has an expansion as a power series in z' in a small disk centered at $z'=0$ and the coefficients of the expansion are determined by its derivatives at $z'=0$. By Lemma 7.7, for each $k\in\mathbb{N}$ the k-th derivative at $z'=0$ of the sum of (9.144) is the sum of the absolutely convergent series

$$\sum_{n\in\mathbb{R}}\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}x_0^p x_1^q x_0^{-1}\delta\Big(\frac{x_1-z_2}{x_0}\Big)n^k\cdot$$

$$\cdot((L'_{P(z_0)}(0)-n)^i(\lambda_n^{(2)}(Y^{\prime o}_4(v,x_1)w'_{(4)},w_{(3)}))))(w_{(1)}\otimes w_{(2)})$$

$$-\sum_{n\in\mathbb{R}}\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}x_0^p x_1^q x_0^{-1}\delta\Big(\frac{z_2-x_1}{-x_0}\Big)n^k\cdot$$

$$\cdot((L'_{P(z_0)}(0)-n)^i(\lambda_n^{(2)}(w'_{(4)},Y_3(v,x_1)w_{(3)}))))(w_{(1)}\otimes w_{(2)})$$

for each $i=0,\ldots,N_{p,q}$. Thus we see that the expansion of the sum of (9.144) as a power series in z' is

$$\sum_{k\in\mathbb{N}}\Big(\sum_{n\in\mathbb{R}}\operatorname{Res}_{x_0}\operatorname{Res}_{x_1}x_0^p x_1^q x_0^{-1}\delta\Big(\frac{x_1-z_2}{x_0}\Big)\frac{n^k}{k!}\cdot$$

$$\cdot ((L'_{P(z_0)}(0) - n)^i (\lambda_n^{(2)}(Y_4'^o(v, x_1)w'_{(4)}, w_{(3)}))) (w_{(1)} \otimes w_{(2)})\Big)(z')^k$$

$$- \sum_{k \in \mathbb{N}} \Big(\sum_{n \in \mathbb{R}} \operatorname{Res}_{x_0} \operatorname{Res}_{x_1} x_0^p x_1^q x_0^{-1} \delta\Big(\frac{z_2 - x_1}{-x_0}\Big) \frac{n^k}{k!} \cdot$$

$$\cdot ((L'_{P(z_0)}(0) - n)^i (\lambda_n^{(2)}(w'_{(4)}, Y_3(v, x_1)w_{(3)}))) (w_{(1)} \otimes w_{(2)})\Big)(z')^k \qquad (9.145)$$

for each $i = 0, \ldots, N_{p,q}$. In particular, the power series obtained by substituting z' for y in the left-hand side of (9.143) is absolutely convergent to the sum of the doubly absolutely convergent series

$$\sum_{n \in \mathbb{R}} \operatorname{Res}_{x_0} \operatorname{Res}_{x_1} x_0^p x_1^q x_0^{-1} \delta\Big(\frac{x_1 - z_2}{x_0}\Big) e^{nz'} \cdot$$

$$\cdot \sum_{i=1}^{N_{p,q}} \frac{(z')^i}{i!} ((L'_{P(z_0)}(0) - n)^i (\lambda_n^{(2)}(Y_4'^o(v, x_1)w'_{(4)}, w_{(3)}))) (w_{(1)} \otimes w_{(2)})$$

$$- \sum_{n \in \mathbb{R}} \operatorname{Res}_{x_0} \operatorname{Res}_{x_1} x_0^p x_1^q x_0^{-1} \delta\Big(\frac{z_2 - x_1}{-x_0}\Big) e^{nz'} \cdot$$

$$\cdot \sum_{i=1}^{N_{p,q}} \frac{(z')^i}{i!} ((L'_{P(z_0)}(0) - n)^i (\lambda_n^{(2)}(w'_{(4)}, Y_3(v, x_1)w_{(3)}))) (w_{(1)} \otimes w_{(2)}) \qquad (9.146)$$

for z' in the small disk above.

We now consider the right-hand side of (9.143) analogously. Since $G(w'_{(4)})$ satisfies the $P^{(2)}(z_0)$-local grading restriction condition, for z' in a neighborhood of $z' = 0$, the series

$$\sum_{n \in \mathbb{R}} (e^{z' L'_{P(z_0)}(0)} \pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}})) (w_{(1)} \otimes w_{(2)}) = \sum_{n \in \mathbb{R}} (e^{z' L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) (w_{(1)} \otimes w_{(2)})$$

(recall (9.113) and (9.114)) is absolutely convergent for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, and for z' in this neighborhood, the sums of these series for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ give an element $\mu^{(2)}_{G(w'_{(4)}), w_{(3)}}(z')$ of $(W_1 \otimes W_2)^*$. Taking

$$\mu = \mu^{(2)}_{G(w'_{(4)}), w_{(3)}}(z')$$

in (9.124), we obtain

$$(Y_{P(z_0)}'^o(v, x_0) \mu^{(2)}_{G(w'_{(4)}), w_{(3)}}(z')) (w_{(1)} \otimes w_{(2)})$$

$$= (\mu^{(2)}_{G(w'_{(4)}), w_{(3)}}(z')) (w_{(1)} \otimes Y_2(v, x_0)w_{(2)})$$

$$+ \operatorname{Res}_{x_2} z_0^{-1} \delta\Big(\frac{x_0 - x_2}{z_0}\Big) (\mu^{(2)}_{G(w'_{(4)}), w_{(3)}}(z')) (Y_1(v, x_2)w_{(1)} \otimes w_{(2)})$$

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$$\begin{aligned}
&= \sum_{n \in \mathbb{R}} (e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) (w_{(1)} \otimes Y_2(v, x_0) w_{(2)}) \\
&\quad + \mathrm{Res}_{x_2} z_0^{-1} \delta\Big(\frac{x_0 - x_2}{z_0}\Big) \sum_{n \in \mathbb{R}} (e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) (Y_1(v, x_2) w_{(1)} \otimes w_{(2)}) \\
&= \sum_{n \in \mathbb{R}} (e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) (w_{(1)} \otimes Y_2(v, x_0) w_{(2)}) \\
&\quad + \mathrm{Res}_{x_2} \sum_{n \in \mathbb{R}} z_0^{-1} \delta\Big(\frac{x_0 - x_2}{z_0}\Big) (e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) (Y_1(v, x_2) w_{(1)} \otimes w_{(2)}),
\end{aligned}$$
(9.147)

where in the second term, the coefficient of each monomial in x_0 involves only the single infinite sum $\sum_{n \in \mathbb{R}}$, so that $\sum_{n \in \mathbb{R}}$ can be switched with $z_0^{-1} \delta\big(\frac{x_0 - x_2}{z_0}\big)$. Thus we obtain

$$\begin{aligned}
&\Bigg(Y'^o_{P(z_0)}(v, x_0) \sum_{n \in \mathbb{R}} (e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) \Bigg) (w_{(1)} \otimes w_{(2)}) \\
&\qquad = \sum_{n \in \mathbb{R}} (Y'^o_{P(z_0)}(v, x_0) e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) (w_{(1)} \otimes w_{(2)}),
\end{aligned}$$
(9.148)

with absolute convergence for the coefficient of each monomial in x_0.

Let
$$\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}(z') \in \overline{W_1 \boxtimes_{P(z_0)} W_2}$$

be the element whose homogeneous components of generalized weight $n \in \mathbb{R}$ are

$$e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)}).$$

In particular, we have
$$\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}(z') = e^{z'L'_{P(z_0)}(0)} \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}.$$
(9.149)

Since $|z_2| > |z_0| > 0$, $|e^{-z'} z_2| > |z_0| > 0$ for $|z'|$ sufficiently small. Then the exact same arguments from (9.126) to (9.130) with z_2 replaced by $e^{-z'} z_2$, v by $e^{-z'L(0)} v$ and $\mu^{(2)}_{G(w'_{(4)}), w_{(3)}}$ by

$$\mu = \mu^{(2)}_{G(w'_{(4)}), w_{(3)}}(z')$$

show that the coefficient of each monomial in x_0 and x_1 in

$$\begin{aligned}
&\sum_{n \in \mathbb{R}} (e^{-z'} z_2)^{-1} \delta\left(\frac{x_1 - x_0}{e^{-z'} z_2}\right) (Y'^o_{P(z_0)}(e^{-z'L(0)} v, x_0) \pi_n(\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}(z'))) (w_{(1)} \otimes w_{(2)}) \\
&= \sum_{n \in \mathbb{R}} (e^{-z'} z_2)^{-1} \delta\left(\frac{x_1 - x_0}{e^{-z'} z_2}\right) (Y'^o_{P(z_0)}(e^{-z'L(0)} v, x_0) e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)})) (w_{(1)} \otimes w_{(2)})
\end{aligned}$$
(9.150)

is doubly absolutely convergent for z' in a neighborhood of $z' = 0$ independent of $w_{(1)}$ and $w_{(2)}$ and that

$$\sum_{n \in \mathbb{R}} (e^{-z'} z_2)^{-1} \delta \left(\frac{x_1 - x_0}{e^{-z'} z_2} \right) \pi_n (Y'^o_{P(z_0)} (e^{-z'L(0)} v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}} (z')) (w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} (e^{-z'} z_2)^{-1} \delta \left(\frac{x_1 - x_0}{e^{-z'} z_2} \right) (Y'^o_{P(z_0)} (e^{-z'L(0)} v, x_0) \pi_n (\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}} (z'))) (w_{(1)} \otimes w_{(2)})$$

$$(9.151)$$

for z' in this same neighborhood, again with double absolute convergence. Replacing x_1 and x_0 in (9.150) and (9.151) by $e^{-z'} x_1$ and $e^{-z'} x_0$, respectively, and dividing both sides by $e^{z'}$, we see that the coefficient of each monomial in x_0 and x_1 in

$$\sum_{n \in \mathbb{R}} z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) (Y'^o_{P(z_0)} (e^{-z'L(0)} v, e^{-z'} x_0) e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)} (w'_{(4)}, w_{(3)})) (w_{(1)} \otimes w_{(2)}) \quad (9.152)$$

is doubly absolutely convergent in the same neighborhood of $z' = 0$ and that

$$\sum_{n \in \mathbb{R}} z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) \pi_n (Y'^o_{P(z_0)} (e^{-z'L(0)} v, e^{-z'} x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}} (z')) (w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) (Y'^o_{P(z_0)} (e^{-z'L(0)} v, e^{-z'} x_0) \pi_n (\widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}} (z'))) (w_{(1)} \otimes w_{(2)})$$

$$(9.153)$$

for z' in this same neighborhood, with double absolute convergence.

Using (9.120), (2.62) and (2.65), we have, as in (3.86) but for Y^o (or by invoking (3.86), (2.73) and (2.75))

$$e^{z'L'_{P(z_0)}(0)} Y'^o_{P(z_0)} (v, x_0) \lambda_n^{(2)} (w'_{(4)}, w_{(3)})$$

$$= Y'^o_{P(z_0)} (e^{-z'L(0)} v, e^{-z'} x_0) e^{z'L'_{P(z_0)}(0)} \lambda_n^{(2)} (w'_{(4)}, w_{(3)}) \quad (9.154)$$

for each $n \in \mathbb{R}$, and so

$$e^{z'L'_{P(z_0)}(0)} Y'^o_{P(z_0)} (v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}$$

$$= Y'^o_{P(z_0)} (e^{-z'L(0)} v, e^{-z'} x_0) e^{z'L'_{P(z_0)}(0)} \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}. \quad (9.155)$$

Also, by definition, $e^{z'L'_{P(z_0)}(0)}$ commutes with π_n for $n \in \mathbb{R}$. From these formulas, we obtain

$$\sum_{n \in \mathbb{R}} z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) (e^{z'L'_{P(z_0)}(0)} \pi_n (Y'^o_{P(z_0)} (v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}})) (w_{(1)} \otimes w_{(2)})$$

$$
= \sum_{n \in \mathbb{R}} z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n\big(e^{z' L'_{P(z_0)}(0)} Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}\big)(w_{(1)} \otimes w_{(2)})
$$

$$
= \sum_{n \in \mathbb{R}} z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n\big(Y'^o_{P(z_0)}(e^{-z' L(0)} v, e^{-z'} x_0) e^{z' L'_{P(z_0)}(0)} \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}\big)(w_{(1)} \otimes w_{(2)})
$$

$$
= \sum_{n \in \mathbb{R}} z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n\big(Y'^o_{P(z_0)}(e^{-z' L(0)} v, e^{-z'} x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}(z')\big)(w_{(1)} \otimes w_{(2)}), \quad (9.156)
$$

with double absolute convergence in the same neighborhood of $z' = 0$ for each coefficient in x_0 and x_1 for each sum, since we know that the right-hand side has double absolute convergence. (We recall that the double sums are over $n \in \mathbb{R}$ and over the integral powers of x_0. The operator $e^{z' L'_{P(z_0)}(0)}$ is applied to the indicated vectors, and in each case, it acts as a convergent sum of operators on a suitable *finite*-dimensional vector space because $W_1 \boxtimes_{P(z_0)} W_2$ is a generalized module.)

From (9.139), (9.140) and (9.142) and this double absolute convergence for the left-hand side of (9.156), we have, for each pair $p, q \in \mathbb{Z}$,

$$
\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n \in \mathbb{R}} z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \big(e^{z' L'_{P(z_0)}(0)} \pi_n(Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}})\big)(w_{(1)} \otimes w_{(2)})
$$

$$
= \mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q \sum_{n \in \mathbb{R}} z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) e^{nz'} \cdot
$$

$$
\cdot \left(\sum_{i=0}^{N_{p,q}} \frac{(z')^i}{i!} \big((L'_{P(z_0)}(0) - n)^i (\pi_n(Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}))\big)(w_{(1)} \otimes w_{(2)}) \right);
$$

$$
(9.157)
$$

on the right-hand side, each summand of the doubly absolutely convergent sum (in the neighborhood of $z' = 0$ above) has been replaced by a finite sum over i.

Since both sides of (9.157) are doubly absolutely convergent, we can write (9.157) with the sums over $n \in \mathbb{R}$ on the outside and the sums over the integral powers of x_0 on the inside:

$$
\sum_{n \in \mathbb{R}} \left(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \cdot \right.
$$

$$
\left. \cdot (e^{z' L'_{P(z_0)}(0)} \pi_n(Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}))(w_{(1)} \otimes w_{(2)}) \right)
$$

$$
= \sum_{n \in \mathbb{R}} \left(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) e^{nz'} \cdot \right.
$$

$$
\left. \cdot \left(\sum_{i=0}^{N_{p,q}} \frac{(z')^i}{i!} \big((L'_{P(z_0)}(0) - n)^i (\pi_n(Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}))\big)(w_{(1)} \otimes w_{(2)}) \right) \right)
$$

$$= \sum_{n \in \mathbb{R}} e^{nz'} \left(\sum_{i=0}^{N_{p,q}} \frac{(z')^i}{i!} \left((L'_{P(z_0)}(0) - n)^i \cdot \right. \right.$$
$$\left. \left. \cdot \left(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) \pi_n (Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}) \right) \right) (w_{(1)} \otimes w_{(2)}) \right),$$

$$(9.158)$$

where the last equality follows from the finiteness of the sum over integral powers of x_0 on the right-hand side of (9.130) for each $n \in \mathbb{R}$; in particular, on the right-hand side of (9.158), for each $n \in \mathbb{R}$ the sum is finite, and thus the same is true of the left-hand side of (9.158), the powers of x_0 entering into the inner sum being the same on the two sides. The outer sums (over $n \in \mathbb{R}$) are of course absolutely convergent in our neighborhood of $z' = 0$, which, we recall, is independent of $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

We again use the part of the proof of Theorem 9.17 from (9.46) to (9.51) with $-l^0(z)$ replaced by z' (as in Remark 9.19 and (9.144); what follows is a variant of the argument in (9.144)–(9.146)): Since the left-hand side of (9.158) is absolutely convergent in a neighborhood of $z' = 0$ independent of $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, from (5.110), for $k \in \mathbb{N}$ the series

$$\sum_{n \in \mathbb{R}} \left(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) \left((L'_{P(z_0)}(0))^k e^{z' L'_{P(z_0)}(0)} \cdot \right. \right.$$
$$\left. \left. \cdot \pi_n (Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}) \right) (w_{(1)} \otimes w_{(2)}) \right)$$
$$= \sum_{n \in \mathbb{R}} \left(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) \left(e^{z' L'_{P(z_0)}(0)} \pi_n (Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}) \right) \cdot \right.$$
$$\left. \cdot \left(\sum_{i=0}^{k} \binom{k}{i} (L(0) + z_0 L(-1))^i w_{(1)} \otimes (L(0))^{k-i} w_{(2)} \right) \right)$$

$$(9.159)$$

obtained by summing the term-by-term k-th derivatives with respect to z' on the left-hand side of (9.158) is absolutely convergent in the same neighborhood. By (9.158), equivalently, the series of term-by-term k-th derivatives with respect to z' for the right-hand side of (9.158) is absolutely convergent (as an iterated series) in the same neighborhood, for $k \in \mathbb{N}$. Thus by Proposition 7.9, for each $i = 0, \ldots, N_{p.q}$,

$$\sum_{n \in \mathbb{R}} e^{nz'} \left((L'_{P(z_0)}(0) - n)^i \cdot \right.$$
$$\left. \cdot \left(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta \left(\frac{x_1 - x_0}{z_2} \right) \pi_n (Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}) \right) \right) (w_{(1)} \otimes w_{(2)})$$

$$(9.160)$$

is absolutely convergent in the same neighborhood. From Lemma 7.7, (9.160) is an analytic function in this neighborhood, and thus so is the right-hand side of (9.158), which equals

the sum of the absolutely convergent double series

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N_{p,q}} e^{nz'} \frac{(z')^i}{i!} \bigg((L'_{P(z_0)}(0) - n)^i \cdot$$

$$\cdot \bigg(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n(Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}) \bigg) \bigg) (w_{(1)} \otimes w_{(2)}).$$

$$(9.161)$$

Moreover, the k-th derivative of (9.161) with respect to with respect to z' at $z' = 0$ is the sum of the absolutely convergent series obtained by setting $z' = 0$ in the left-hand side of (9.159), namely,

$$\sum_{n \in \mathbb{R}} \mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) ((L'_{P(z_0)}(0))^k \pi_n(Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}})) (w_{(1)} \otimes w_{(2)}).$$

$$(9.162)$$

This information determines a power series expansion of the analytic function (9.161) in a small disk centered at $z' = 0$, and in fact we know that it is obtained as follows: The right-hand side of (9.143), which is a formal power series in y whose coefficients are absolutely convergent sums over $n \in \mathbb{R}$, is obtained by applying $\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q$ to the right-hand side of (9.138), and the coefficient of $y^k/k!$ in this formal power series is exactly (9.162). Hence in a small disk centered at $z' = 0$, the substitution of z' for y in the right-hand side of (9.143) gives a convergent power series expansion of the analytic function (9.161), or equivalently, (9.158).

We have shown that the left- and right-hand sides of (9.143) with y replaced by z' are absolutely convergent to the sums of (9.146) and of (9.161), respectively, for z' in small disks centered at $z' = 0$. Thus the analytic functions (9.146) and (9.161) must be equal in the intersection of these disks. That is, for $p, q \in \mathbb{Z}$,

$$\sum_{n \in \mathbb{R}} \sum_{i=0}^{N_{p,q}} e^{nz'} \frac{(z')^i}{i!} \mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q x_0^{-1} \delta\left(\frac{x_1 - z_2}{x_0}\right) \cdot$$

$$\cdot ((L'_{P(z_0)}(0) - n)^i (\lambda_n^{(2)}(Y'^o_4(v, x_1) w'_{(4)}, w_{(3)}))) (w_{(1)} \otimes w_{(2)})$$

$$- \sum_{n \in \mathbb{R}} \sum_{i=0}^{N_{p,q}} e^{nz'} \frac{(z')^i}{i!} \mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q x_0^{-1} \delta\left(\frac{z_2 - x_1}{-x_0}\right) \cdot$$

$$\cdot ((L'_{P(z_0)}(0) - n)^i (\lambda_n^{(2)}(w'_{(4)}, Y_3(v, x_1) w_{(3)}))) (w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} \sum_{i=0}^{N_{p,q}} e^{nz'} \frac{(z')^i}{i!} \bigg((L'_{P(z_0)}(0) - n)^i \cdot$$

$$\cdot \bigg(\mathrm{Res}_{x_0} \mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n(Y'^o_{P(z_0)}(v, x_0) \widetilde{\mu}^{(2)}_{G(w'_{(4)}), w_{(3)}}) \bigg) \bigg) (w_{(1)} \otimes w_{(2)}),$$

$$(9.163)$$

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with double absolute convergence for z' in a small disk centered at $z' = 0$.

We can now apply Proposition 7.8. Since $\mathbb{R} \times \{0, \ldots, N_{p,q}\}$ is a unique expansion set for any $p, q \in \mathbb{Z}$, we obtain from (9.163), taking $i = 0$, that

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q x_0^{-1} \delta\Big(\frac{x_1 - z_2}{x_0}\Big) (\lambda_n^{(2)}(Y_4'^o(v, x_1)w_{(4)}', w_{(3)}))(w_{(1)} \otimes w_{(2)})$$

$$-\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q x_0^{-1} \delta\Big(\frac{z_2 - x_1}{-x_0}\Big) (\lambda_n^{(2)}(w_{(4)}', Y_3(v, x_1)w_{(3)}))(w_{(1)} \otimes w_{(2)})$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_0^p x_1^q z_2^{-1} \delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n(Y_{P(z_0)}'^o(v, x_0)\widetilde{\mu}_{G(w_{(4)}'), w_{(3)}}^{(2)})(w_{(1)} \otimes w_{(2)})$$

for $n \in \mathbb{R}$ and $p, q \in \mathbb{Z}$, or equivalently, that

$$x_0^{-1}\delta\Big(\frac{x_1 - z_2}{x_0}\Big)(\lambda_n^{(2)}(Y_4'^o(v, x_1)w_{(4)}', w_{(3)}))(w_{(1)} \otimes w_{(2)})$$

$$-x_0^{-1}\delta\Big(\frac{z_2 - x_1}{-x_0}\Big)(\lambda_n^{(2)}(w_{(4)}', Y_3(v, x_1)w_{(3)}))(w_{(1)} \otimes w_{(2)})$$

$$= z_2^{-1}\delta\left(\frac{x_1 - x_0}{z_2}\right) \pi_n(Y_{P(z_0)}'^o(v, x_0)\widetilde{\mu}_{G(w_{(4)}'), w_{(3)}}^{(2)})(w_{(1)} \otimes w_{(2)})$$

for $n \in \mathbb{R}$.

We have thus proved the equality of (9.122) and (9.123), and this proves (9.117) and hence (9.116).

Similarly, one can prove that $\widetilde{G}$ intertwines the corresponding $\mathfrak{sl}(2)$ actions; in fact, the appropriate argument arises from setting $v = \omega$ above, and taking the relevant three components at each step. $\quad\square$

Recalling the assumptions given before Lemma 9.22, we see that the map $\widetilde{G}$ is $\widetilde{A}$-compatible (recall Definition 5.16 and (5.88)), from the definitions, Remark 9.11 and Proposition 9.12. Thus by Lemma 9.22 and Proposition 5.60 there is a unique $Q(z_2)$-intertwining map $\widetilde{I}$ of type $\binom{W_1 \boxtimes_{P(z_0)} W_2}{W_4' \ W_3}$ such that

$$\widetilde{G}(w)(w_{(4)}' \otimes w_3) = \langle w, \widetilde{I}(w_{(4)}' \otimes w_{(3)}) \rangle$$

for $w \in W_1 \boxtimes_{P(z_0)} W_2$, $w_{(4)}' \in W_4'$ and $w_{(3)} \in W_3$. By Corollary 4.42, there exists a unique $P(z_2)$-intertwining map I of type $\binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2 \ W_3}$ such that

$$\langle w, \widetilde{I}(w_{(4)}' \otimes w_{(3)}) \rangle = \langle w_{(4)}', I(w \otimes w_{(3)}) \rangle,$$

or equivalently,

$$\widetilde{G}(w)(w_{(4)}' \otimes w_3) = \langle w_{(4)}', I(w \otimes w_{(3)}) \rangle,$$

or equivalently (by (9.115)),

$$\langle w, \widetilde{\mu}_{G(w_{(4)}'), w_{(3)}}^{(2)} \rangle_{W_1 \boxtimes_{P(z_0)} W_2} = \langle w_{(4)}', I(w \otimes w_{(3)}) \rangle$$

for $w \in W_1 \boxtimes_{P(z_0)} W_2$, $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$. Taking

$$w = \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{R}$ and invoking Proposition 4.23, we see that I is unique such that

$$\langle \pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}), \widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}} \rangle_{W_1 \boxtimes_{P(z_0)} W_2} = \langle w'_{(4)}, I(\pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle,$$

or equivalently (by (9.118)),

$$(\lambda_n^{(2)}(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) = \langle w'_{(4)}, I(\pi_n(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle \tag{9.164}$$

for all $w_{(j)} \in W_j$ and $w'_{(4)} \in W'_4$.

Now we sum (9.164) over $n \in \mathbb{R}$ to obtain the equality

$$\widetilde{\mu}^{(2)}_{G(w'_{(4)}),w_{(3)}}(w_{(1)} \otimes w_{(2)}) = \langle w'_{(4)}, I((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle$$

of absolutely convergent sums; for the left-hand side we recall (9.109) and (9.110) and for the right-hand side we invoke the convergence condition for intertwining maps in $\mathcal{C}$ for the $P(z_0)$-intertwining map $\boxtimes_{P(z_0)}$. That is, from Definition 7.1, Remarks 8.12 and 9.11 and Definition 9.1,

$$\begin{aligned}
\langle w'_{(4)}, &I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle \\
&= \langle w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle \\
&= (G(w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&= \langle w'_{(4)}, I((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle.
\end{aligned}$$

Moreover, by Proposition 8.19, this equality,

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle = \langle w'_{(4)}, I((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle \tag{9.165}$$

for all $w_{(j)} \in W_j$ and $w'_{(4)} \in W'_4$ determines the $P(z_2)$-intertwining map I uniquely.

We have now proved Part 1 of the theorem below, which states in particular that under the assumptions above, this product of intertwining maps can be written as an iterate of certain intertwining maps. Moreover, as is guaranteed by Proposition 8.19 and proved directly above, the intermediate module can always be taken as $W_1 \boxtimes_{P(z_0)} W_2$. Part 2 of this theorem is proved analogously.

Theorem 9.23 *Assume that $\mathcal{C}$ is closed under images, that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that*

$$|z_1| > |z_2| > |z_0| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$. Assume also that $W_1 \boxtimes_{P(z_0)} W_2$ and $W_2 \boxtimes_{P(z_2)} W_3$ exist in $\mathcal{C}$.

1. *Let I_1 and I_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively. Suppose that for each $w'_{(4)} \in W'_4$,*

$$\lambda = (I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ be the (unique) series weakly absolutely convergent to $\mu^{(2)}_{\lambda, w_{(3)}}$ as indicated in the $P^{(2)}(z_0)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, the generalized V-submodule of the generalized V-module $W^{(2)}_{\lambda, w_{(3)}}$ (given by Theorem 9.17) generated by $\lambda_n^{(2)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$. Then the product

$$I_1 \circ (1_{W_1} \otimes I_2)$$

can be expressed as an iterate, and in fact, there exists a unique $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2 \ W_3}$ such that

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)}))\rangle = \langle w'_{(4)}, I^1((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)})\rangle$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

2. *Analogously, let I^1 and I^2 be $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Suppose that for each $w'_{(4)} \in W'_4$,*

$$\lambda = (I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ be the (unique) series weakly absolutely convergent to $\mu^{(1)}_{\lambda, w_{(1)}}$ as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, the generalized V-submodule of the generalized V-module $W^{(1)}_{\lambda, w_{(1)}}$ (given by Theorem 9.17) generated by $\lambda_n^{(1)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$. Then the iterate

$$I^1 \circ (I^2 \otimes 1_{W_3})$$

can be expressed as a product, and in fact, there exists a unique $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 \ W_2 \boxtimes_{P(z_2)} W_3}$ such that

$$\langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)})\rangle = \langle w'_{(4)}, I_1(w_{(1)} \otimes (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))\rangle$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. $\quad\square$

We know from Section 4, in particular, Proposition 4.8, in which we shall take $p = 0$, that $P(z)$-intertwining maps are equivalent to suitable evaluations of logarithmic intertwining operators or ordinary intertwining operators at z. Thus Theorem 9.23 in fact says that under all of the assumptions in the theorem, the following associativity of logarithmic and of ordinary intertwining operators holds:

Corollary 9.24 *Assume that $\mathcal{C}$ is closed under images, that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that*

$$|z_1| > |z_2| > |z_0| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$. Assume also that $W_1 \boxtimes_{P(z_0)} W_2$ and $W_2 \boxtimes_{P(z_2)} W_3$ exist in $\mathcal{C}$.

1. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively. Suppose that for each $w'_{(4)} \in W'_4$, the element $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ given by*

$$\lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\, x_2=z_2}$$

   (recalling (7.14)) for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ satisfies the $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ be the (unique) series weakly absolutely convergent to $\mu_{\lambda, w_{(3)}}^{(2)}$ as indicated in the $P^{(2)}(z_0)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, the generalized V-submodule of the generalized V-module $W_{\lambda, w_{(3)}}^{(2)}$ (given by Theorem 9.17) generated by $\lambda_n^{(2)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$. Then there exists a unique logarithmic intertwining operator (a unique ordinary intertwining operator in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ of type $\binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2\ W_3}$ such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\, x_2=z_2}$$
$$= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}_{\boxtimes_{P(z_0)}, 0}(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)})\rangle\Big|_{x_0=z_0,\, x_2=z_2} \tag{9.166}$$

   (recalling (4.18) and (7.13)) for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. In particular, the product of the logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ and $\mathcal{Y}_2$ evaluated at z_1 and z_2, respectively, can be expressed as an iterate (with the intermediate generalized V-module $W_1 \boxtimes_{P(z_0)} W_2$) of logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) evaluated at z_2 and z_0.

2. *Analogously, let $\mathcal{Y}^1$ and $\mathcal{Y}^2$ be logarithmic intertwining operators (ordinary intertwin-*
   ing operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively.
   Suppose that for each $w'_{(4)} \in W'_4$, the element $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ given by

$$\lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)} \rangle \Big|_{x_0=z_0,\, x_2=z_2}$$

(recalling (7.13)) satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ be the (unique) series weakly absolutely convergent to $\mu_{\lambda,w_{(1)}}^{(1)}$ as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, the generalized V-submodule of the generalized V-module $W_{\lambda,w_{(1)}}^{(1)}$ (given by Theorem 9.17) generated by $\lambda_n^{(1)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$. Then there exists a unique logarithmic intertwining operator (a unique ordinary intertwining operator in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ of type $\binom{W_4}{W_1 \ W_2 \boxtimes_{P(z_2)} W_3}$ such that

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)} \rangle \Big|_{x_0=z_0,\, x_2=z_2}$$

$$= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, x_2)w_{(3)} \rangle \Big|_{x_1=z_1,\, x_2=z_2} \tag{9.167}$$

(again recalling (4.18) and (7.14)) for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. In particular, the iterate of the logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ and $\mathcal{Y}^2$ evaluated at z_2 and z_0, respectively, can be expressed as a product (with the intermediate generalized V-module $W_2 \boxtimes_{P(z_2)} W_3$) of logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) evaluated at z_1 and z_2.

Proof We prove only Part 1, the proof of Part 2 being analogous.

By (4.15), we have

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)} \rangle \Big|_{x_1=z_1,\, x_2=z_2} = \langle w'_{(4)}, I_{\mathcal{Y}_1,0}(w_{(1)} \otimes I_{\mathcal{Y}_2,0}(w_{(2)} \otimes w_{(3)})) \rangle$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. By Part 1 of Theorem 9.23, There exists a unique $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2 \ W_3}$ such that

$$\langle w'_{(4)}, I_{\mathcal{Y}_1,0}(w_{(1)} \otimes I_{\mathcal{Y}_2,0}(w_{(2)} \otimes w_{(3)})) \rangle = \langle w'_{(4)}, I^1((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle.$$

By Proposition 4.8, we have

$$\langle w'_{(4)}, I^1((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle = \langle w'_{(4)}, \mathcal{Y}_{I^1,0}(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}) \rangle \Big|_{x_0=z_0,\, x_2=z_2}.$$

Taking $\mathcal{Y}^1 = \mathcal{Y}_{I^1,0}$, we obtain (9.166). Since I^1 is unique, $\mathcal{Y}^1 = \mathcal{Y}_{I^1,0}$ is also unique, by Proposition 4.8 (as in Corollary 8.20). In the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$, $\mathcal{Y}^1$ is an ordinary intertwining operator, by Remark 3.23. $\quad\square$

Theorem 9.23 and Corollary 9.24 both have two parts, each with a major assumption, involving the $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition) in Part 1 and the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition) in Part 2, and the resulting pair of assumptions essentially form most of what we will call the "expansion condition" (see Definition 9.28 below). We would now like to show that these two major assumptions are actually equivalent to each other. For this, we need the equivalence (Theorem 9.26 below) of two versions of the associativity of logarithmic or ordinary intertwining operators, namely, that every product can be expressed as an iterate, and on the other hand, that every iterate can be expressed as a product (recall the conclusions of Part 1 and Part 2 of Corollary 9.24). Theorem 9.26 and the lemma below used in its proof do not use any results in Section 8 or any of the results in the present section that we have obtained so far.

Recall that in Section 7 we proved two formulas, (7.6) and (7.9), using the maps Ω_r (recall (3.77)), on writing products of intertwining operators satisfying certain conditions in terms of iterates, and vice versa. In the next lemma, we shall use (7.6) and (7.9) to prove analogues of these two formulas. In the statement and proof of this lemma, we shall use the analyticity, Proposition 7.14, and Proposition 7.20 and Remark 7.21 to rewrite the consequences (7.7) and (7.10) of (7.6) and (7.9), respectively, and to write analogous expressions.

Lemma 9.25 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds. Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$. Then:*

1. *For any nonzero complex numbers z_1, z_2 such that*

$$|z_1| > |z_0| > 0, \quad |z_2| > |z_0| > 0$$

   (with $z_0 = z_1 - z_2$ as usual), there exist $p, q \in \mathbb{Z}$ such that for any logarithmic (in particular, ordinary) intertwining operators $\mathcal{Y}^1$ and $\mathcal{Y}^2$ of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, we have

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=z_2}$$
$$= \langle e^{z_1 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, y_1)\Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, y_2) \cdot$$
$$\cdot w_{(1)}\rangle_{W_4}\Big|_{y_1^n=e^{nl_p(-z_1)},\, \log y_1=l_p(-z_1),\, y_2^n=e^{nl_q(-z_0)},\, \log y_2=l_q(-z_0)}$$

$$(9.168)$$

   for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

2. *For any nonzero complex numbers z_1, z_2 such that*

$$|z_1| > |z_2| > 0, \quad |z_0| > |z_2| > 0,$$

there exist $\tilde{p}, \tilde{q} \in \mathbb{Z}$ such that for any logarithmic (in particular, ordinary) intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, we have

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_1=z_1,\, x_2=z_2}$$
$$= \langle e^{z_1 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(\Omega_{-1}(\mathcal{Y}_2)(w_{(3)}, y_0)w_{(2)}, y_2) \cdot$$
$$\cdot w_{(1)}\rangle_{W_4}\Big|_{y_0^n=e^{nl_{\tilde{p}}(-z_2)},\, \log y_0=l_{\tilde{p}}(-z_2),\, y_2^n=e^{nl_{\tilde{q}}(-z_0)},\, \log y_2=l_{\tilde{q}}(-z_0)}$$

(9.169)

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

Proof We prove only (9.168); (9.169) is proved similarly, and at the end of the proof we discuss it briefly. In the first part of our proof, we shall interpret substitution notation such as "$x_2 = e^{-i\pi}z_2$" the same way we did in the proof of Proposition 7.3, namely (in this instance), as the substitution of

$$e^{\log z_2 - \pi i}$$

for x_2 (rather than as in (7.13) and (7.14), where $p = 0$); more precisely, for convenience we shall reverse the occurrences of Ω_0 and Ω_{-1} in (7.6)–(7.7), and correspondingly, we shall use $x_2 = e^{+i\pi}z_2$, which serves to replace x_2 by $e^{\log z_2 + \pi i}$.

Using the formulas (7.6)–(7.7) (or more precisely, the indicated variant of (7.7)), along with Proposition 3.44 and (3.60), we have

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=z_2}$$
$$= \langle e^{z_2 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, x_2)\Omega_0(\Omega_{-1}(\mathcal{Y}^2))(w_{(1)}, x_0)w_{(2)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=e^{\pi i}z_2}$$
$$= \langle e^{z_2 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, x_2)e^{x_0 L(-1)}\Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, e^{\pi i}x_0)w_{(1)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=e^{\pi i}z_2}$$
$$= \langle e^{z_2 L'(1)}w'_{(4)}, e^{x_0 L(-1)}\Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, x_2 - x_0)\Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, e^{\pi i}x_0)w_{(1)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=e^{\pi i}z_2}$$
$$= \langle e^{x_0 L'(1)}e^{z_2 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, x_2 - x_0)\Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, e^{\pi i}x_0)w_{(1)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=e^{\pi i}z_2}$$
$$= \langle e^{z_0 L'(1)}e^{z_2 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, x_2 + x_0)\Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, x_0) \cdot$$
$$\cdot w_{(1)}\rangle_{W_4}\Big|_{x_0^n=e^{nl_q(-z_0)},\, \log x_0=l_q(-z_0),\, x_2^n=e^{nl_{\tilde{p}}(-z_2)},\, \log x_2=l_{\tilde{p}}(-z_2)},$$

(9.170)

for some $\tilde{p}, q \in \mathbb{Z}$ independent of $\mathcal{Y}^1$, $\mathcal{Y}^2$, $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$ (see the discussion before (7.7)). Note that the left-hand side of (9.170), as a multisum obtained by substituting powers of the formal variables x_0, x_2, $\log x_0$ and $\log x_2$ by the indicated complex numbers, is absolutely convergent by Proposition 7.20, and each step in (9.170) means that the multisums on both sides are both absolutely convergent and are equal. In particular, the right-hand

side, as a multisum obtained by substituting the powers of the formal variables x_0, x_2, $\log x_0$ and $\log x_2$ by the indicated complex numbers, is absolutely convergent.

Note that since $e^{\pm z_1 L'(1)}$ are linear automorphisms of W'_4, W'_4 is spanned by homogeneous elements of the form $e^{z_1 L'(1)} w'_{(4)}$. We need only prove (9.168) for homogeneous $w_{(1)}, w_{(2)}, w_{(3)}$ and $e^{z_1 L'(1)} w'_{(4)}$, and we assume this homogeneity. Recalling Proposition 7.20 and (7.46), let

$$\Delta = -\mathrm{wt}\ e^{z_1 L'(1)} w'_{(4)} + \mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + \mathrm{wt}\ w_{(3)} \in \mathbb{R}$$

and define

$$a_{n,j,i} = \langle e^{z_1 L'(1)} w'_{(4)}, (w_{(3)})^{\Omega_{-1}(\mathcal{Y}^1)}_{\Delta-n-2,j} (w_{(2)})^{\Omega_{-1}(\mathcal{Y}^2)}_{n,i} w_{(1)} \rangle \in \mathbb{C}$$

for $n \in \mathbb{R}$, $j = 0, \dots, M$, $i = 0, \dots, N$. From this expression of $a_{n,j,i}$ and (3.25), for $\mu \in \mathbb{R}/\mathbb{Z}$, there exists $R_\mu \in \mu$ such that $a_{n,j,i} = 0$ for any $n \in \mu$ with $n > R_\mu$. Then since

$$\langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, x_2 + x_0) \Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, x_0) w_{(1)} \rangle_{W_4}$$

$$= \sum_{n \in \mathbb{R}} \sum_{j=0}^{M} \sum_{i=0}^{N} a_{n,j,i} (x_2 + x_0)^{-\Delta+n+1} (\log(x_2 + x_0))^j x_0^{-n-1} (\log x_0)^i$$

$$= \sum_{m \in \mathbb{R}} \sum_{j=0}^{M} \sum_{i=0}^{N} \left(\sum_{k \in \mathbb{N}} a_{-m-1+k,j,i} \binom{-\Delta-m+k}{k} x_2^{-\Delta-m} x_0^m \right) \cdot$$

$$\cdot \left(\log x_2 + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{x_0^l}{x_2^l} \right)^j (\log x_0)^i,$$

the right-hand side of (9.170) is equal to

$$\sum_{m \in \mathbb{R}} \sum_{j=0}^{M} \sum_{i=0}^{N} \left(\sum_{k \in \mathbb{N}} a_{-m-1+k,j,i} \binom{-\Delta-m+k}{k} e^{(-\Delta-m) l_{\tilde{p}}(-z_2)} e^{m l_q(-z_0)} \right) \cdot$$

$$\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i, \qquad (9.171)$$

an absolutely convergent triple sum since $|z_2| > |z_0| > 0$, with the first of the inner sums finite (since $a_{-m-1+k,j,i} = 0$ for $k > m+1+R_{\overline{-m}}$ where $\overline{-m}$ is the congruence class of $-m$) and the second of the inner sums absolutely convergent, again since $|z_2| > |z_0| > 0$.

Since

$$l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l}$$

is a value of the multivalued logarithmic function at the point $-z_1$, there exists $p \in \mathbb{Z}$ such that

$$l_p(-z_1) = l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l}.$$

Note that p is independent of $\mathcal{Y}^1$, $\mathcal{Y}^2$, $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$. Then since $|z_1| > |z_0| > 0$, Proposition 7.20 and (7.46) give that the right-hand side of (9.168) with p and q as above is equal to the absolutely convergent triple sum

$$
\sum_{n \in \mathbb{R}} \sum_{j=0}^{M} \sum_{i=0}^{N} a_{n,j,i} e^{(-\Delta+n+1)l_p(-z_1)} l_p(-z_1)^j e^{(-n-1)l_q(-z_0)} l_q(-z_0)^i
$$

$$
= \sum_{n \in \mathbb{R}} \sum_{j=0}^{M} \sum_{i=0}^{N} \left(\sum_{k \in \mathbb{N}} a_{n,j,i} \binom{-\Delta+n+1}{k} e^{(-\Delta+n-k+1)l_{\tilde{p}}(-z_2)} e^{(-n-1+k)l_q(-z_0)} \right) \cdot
$$

$$
\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i, \qquad (9.172)
$$

with the inner sums absolutely convergent binomial and logarithmic series since $|z_2| > |z_0| > 0$.

We now consider complex variables z'_1, z'_2 and $z'_0 = z'_1 - z'_2$. We view z'_2 and z'_0 as independent variables. Let U be any open subset of the region $|z'_2 + z'_0| > |z_0|$ and $|z'_2| > |z'_0|$ of $\mathbb{C}^2$ such that its projection U_2 to the z'_2 coordinate is simply connected and let $l(-z'_2)$ be any single-valued analytic branch of the logarithmic function of $-z'_2$ defined for $z'_2 \in U_2$. Then

$$
\tilde{l}(-z'_1) = l(-z'_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z'_0)^l}{(-z'_2)^l}
$$

is a single-valued analytic branch of the logarithmic function of $-z'_1$ for $z'_1 \in z'_0 + U_2$. By Proposition 7.20, (7.46) and Proposition 7.14,

$$
\langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, y_1) \Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, y_2) \cdot
$$

$$
\cdot w_{(1)} \rangle_{W_4} \Big|_{y_1^n = e^{n\tilde{l}(-z'_1)},\ \log y_1 = \tilde{l}(-z'_1),\ y_2^n = e^{nl_q(-z_0)},\ \log y_2 = l_q(-z_0)}
$$

$$
= \sum_{n \in \mathbb{R}} \sum_{j=0}^{M} \sum_{i=0}^{N} a_{n,j,i} e^{(-\Delta+n+1)\tilde{l}(-z'_1)} \tilde{l}(-z'_1)^j e^{(-n-1)l_q(-z_0)} l_q(-z_0)^i \qquad (9.173)
$$

and the corresponding series of its derivatives are absolutely convergent as triple sums when $(z'_2, z'_0) \in U$. By Proposition 7.9 (see also Corollary 7.10 and its proof), for each $j = 0, \dots, M$ and $i = 0, \dots, N$,

$$
\sum_{n \in \mathbb{R}} a_{n,j,i} e^{(-\Delta+n+1)\tilde{l}(-z'_1)} e^{(-n-1)l_q(-z_0)} \qquad (9.174)
$$

is absolutely convergent and by Lemma 7.7 is analytic in z'_1 for $z'_1 \in z'_0 + U_2$. Expanding

$$
e^{(-\Delta+n+1)\tilde{l}(-z'_1)} = e^{(-\Delta+n+1)\tilde{l}(-z'_2-z'_0)}
$$

for $n \in \mathbb{R}$ as a power series in z'_0 (as we did above with z_1, z_2 and z_0 in (9.172)), and recalling that for $\mu \in \mathbb{R}/\mathbb{Z}$, there exists $R_\mu \in \mu$ such that $a_{n,j,i} = 0$ for any $n \in \mu$ with $n > R_\mu$, we see

that (9.174) is equal to the absolutely convergent double sum

$$\sum_{\mu \in \mathbb{R}/\mathbb{Z}} \sum_{\tilde{n} \in -\mathbb{N}} \left(\sum_{k \in \mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta+\tilde{n}+R_\mu+1}{k} \right. \cdot$$

$$\left. \cdot e^{(-\Delta+\tilde{n}+R_\mu-k+1)l(-z_2')}(-z_0')^k e^{(-\tilde{n}-R_\mu-1)l_q(-z_0)} \right),$$

$$(9.175)$$

for $(z_2', z_0') \in U$, with the inner sum absolutely convergent since $|z_2'| > |z_0'|$. In particular, as the quotient by $e^{(-\Delta+R_\mu+1)l(-z_2')}e^{(-R_\mu-1)l_q(-z_0)}$ of a subsum of (9.175), the series

$$\sum_{\tilde{n} \in -\mathbb{N}} \left(\sum_{k \in \mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta+\tilde{n}+R_\mu+1}{k} (-z_2')^{\tilde{n}-k}(-z_0')^k(-z_0)^{-\tilde{n}} \right) \qquad (9.176)$$

for each $\mu \in \mathbb{R}/\mathbb{Z}$, $j = 0, \ldots, M$ and $i = 0, \ldots, N$ is absolutely convergent for $(z_2', z_0') \in U$. The sum of (9.175) as the composition of the analytic functions (9.174) and $-z_1' = -z_2' - z_0'$ is analytic in each of z_2' and z_0' for $(z_2', z_0') \in U$, and, by Lemma 7.7, its derivatives with respect to z_2' and z_0' are sums of absolutely convergent series obtained by taking the derivatives term by term. In particular, for each $\mu \in \mathbb{R}/\mathbb{Z}$, $j = 0, \ldots, M$ and $i = 0, \ldots, N$, since $e^{(-\Delta+R_\mu+1)l(-z_2')}e^{(-R_\mu-1)l_q(-z_0)}$ is analytic in z_2' for $z_2' \in U_2$, the sum of (9.176) as the quotient by $e^{(-\Delta+R_\mu+1)l(-z_2')}e^{(-R_\mu-1)l_q(-z_0)}$ of a subsum of (9.175) is analytic in each of z_2' and z_0' for $(z_2', z_0') \in U$ and its derivatives are sums of absolutely convergent series obtained by taking the derivatives term by term. Since U is an arbitrary open subset of the region given by $|z_2' + z_0'| > |z_0|$ and $|z_2'| > |z_0'|$, the sum of (9.176) is analytic with respect to each of z_2' and z_0' in the region $|z_2' + z_0'| > |z_0|$ and $|z_2'| > |z_0'|$.

We now view (9.176) as an analytic function of $(z_2')^{-1}$ and z_0'. The function (9.176) is equal to

$$\sum_{\tilde{n} \in -\mathbb{N}} a_{\tilde{n}+R_\mu,j,i}(-z_2')^{\tilde{n}}(1+(-z_0')(-z_2')^{-1})^{-\Delta+\tilde{n}+R_\mu+1}(-z_0)^{-\tilde{n}}$$

$$= (1+(-z_0')(-z_2')^{-1})^{-\Delta+R_\mu+1} \sum_{\tilde{n} \in -\mathbb{N}} a_{\tilde{n}+R_\mu,j,i}((-z_2')^{-1}(1+(-z_0')(-z_2')^{-1})^{-1}(-z_0))^{-\tilde{n}}.$$

Since the left-hand side is absolutely convergent when $|z_2' + z_0'| > |z_0|$ and $|z_2'| > |z_0'|$, the series

$$\sum_{\tilde{n} \in -\mathbb{N}} a_{\tilde{n}+R_\mu,j,i}((-z_2')^{-1}(1+(-z_0')(-z_2')^{-1})^{-1}(-z_0))^{-\tilde{n}} \qquad (9.177)$$

is also absolutely convergent in the same region. Consider the power series $\sum_{\tilde{n} \in -\mathbb{N}} a_{\tilde{n}+R_\mu,j,i}z^{-\tilde{n}}$. From the discussion above, its radius of convergence is not 0. In particular,

$$\lim_{z \to 0} \sum_{\tilde{n} \in -\mathbb{N}} a_{\tilde{n}+R_\mu,j,i}z^{-\tilde{n}}$$

exists and is equal to $a_{R_\mu,j,i}$. Since the limit of $(-z_2')^{-1}(1+(-z_0')(-z_2')^{-1})^{-1}(-z_0)$ as $(z_2')^{-1}$ approaches 0 is 0, the limit of (9.177) as $(z_2')^{-1}$ approaches 0 is $a_{R_\mu,j,i}$. Thus for fixed $z_0' \in \mathbb{C}$, since the limit of $(1+(-z_0')(-z_2')^{-1})^{-\Delta+R_\mu+1}$ as $(z_2')^{-1}$ approaches 0 is 1, the limit of (9.176) as $(z_2')^{-1}$ approaches 0 is $a_{R_\mu,j,i}$. Hence for fixed $z_0' \in \mathbb{C}$, the singularity $(z_2')^{-1} = 0$ in (9.176) is removable. We know that (9.176) is analytic in z_0' for fixed $(z_2')^{-1} \neq 0$, in our region. Since the limit of the function (9.176) as $(z_2')^{-1}$ approaches 0 is $a_{R_\mu,j,i}$, this function is also analytic in z_0' when $(z_2')^{-1} = 0$. Hence by Hartogs' theorem (see, for example, page 8 of [Sh]), this function is analytic as a function of the two variables $(z_2')^{-1}$ and z_0' in the neighborhood of $(0,0)$ given by $|1 + z_0'(z_2')^{-1}| > |z_0(z_2')^{-1}|$ and $1 > |z_0'(z_2')^{-1}|$.

Let r be a real number satisfying $r > 2|z_0|$. Then for $(z_2')^{-1}$ and z_0' satisfying $|(z_2')^{-1}| < r^{-1}$ and $|z_0'| < r - |z_0|$, we have

$$|z_0'(z_2')^{-1}| < (r - |z_0|)r^{-1} = 1 - |z_0|r^{-1} < 1$$

and

$$|1 + z_0'(z_2')^{-1}| \geq 1 - |z_0'(z_2')^{-1}| > 1 - (r - |z_0|)r^{-1} = |z_0|r^{-1} > |z_0(z_2')^{-1}|.$$

Thus the polydisk given by $|(z_2')^{-1}| < r^{-1}$ and $|z_0'| < r - |z_0|$ is in the region given by $|1 + z_0'(z_2')^{-1}| > |z_0(z_2')^{-1}|$ and $1 > |z_0'(z_2')^{-1}|$. In particular, our function has a power series expansion in $(z_2')^{-1}$ and z_0' and the power series is doubly absolutely convergent in the polydisk. Since the derivatives of this analytic function are obtained by taking the derivatives of the series (9.176) term by term, we see that the power series expansion of this analytic function is the double series

$$\sum_{\tilde{n} \in -\mathbb{N}} \sum_{k \in \mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta + \tilde{n} + R_\mu + 1}{k} (-z_2')^{\tilde{n}-k}(-z_0')^k(-z_0)^{-\tilde{n}}. \tag{9.178}$$

Thus the two iterated series, (9.176) and

$$\sum_{k \in \mathbb{N}} \left(\sum_{\tilde{n} \in -\mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta + \tilde{n} + R_\mu + 1}{k} (-z_2')^{\tilde{n}-k}(-z_0')^k(-z_0)^{-\tilde{n}} \right), \tag{9.179}$$

associated to (9.178) are also absolutely convergent in the polydisk and their sums are equal to the double sum of (9.178) in the polydisk.

Also, $a_{-\tilde{m}+R_\mu-1+k,j,i} = 0$ when $-\tilde{m} - 1 + k > 0$. Thus in the polydisk, we obtain

$$\sum_{\tilde{n} \in -\mathbb{N}} \left(\sum_{k \in \mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta + \tilde{n} + R_\mu + 1}{k} (-z_2')^{\tilde{n}-k}(-z_0')^k(-z_0)^{-\tilde{n}} \right)$$

$$= \sum_{k \in \mathbb{N}} \sum_{\tilde{n} \in -\mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta + \tilde{n} + R_\mu + 1}{k} (-z_2')^{\tilde{n}-k}(-z_0')^k(-z_0)^{-\tilde{n}}$$

$$= \sum_{k \in \mathbb{N}} \sum_{\tilde{m} \in \mathbb{N}-1+k} a_{-\tilde{m}+R_\mu-1+k,j,i} \binom{-\Delta - \tilde{m} + R_\mu + k}{k} (-z_2')^{-\tilde{m}-1}(-z_0')^k(-z_0)^{\tilde{m}+1-k}$$

$$= \sum_{k \in \mathbb{N}} \sum_{\tilde{m} \in \mathbb{N}-1} a_{-\tilde{m}+R_\mu-1+k,j,i} \binom{-\Delta-\tilde{m}+R_\mu+k}{k} (-z_2')^{-\tilde{m}-1}(-z_0')^k(-z_0)^{\tilde{m}+1-k}$$

$$= \sum_{\tilde{m} \in \mathbb{N}-1} \sum_{k \in \mathbb{N}} a_{-\tilde{m}+R_\mu-1+k,j,i} \binom{-\Delta-\tilde{m}+R_\mu+k}{k} (-z_2')^{-\tilde{m}-1}(-z_0')^k(-z_0)^{\tilde{m}+1-k}$$

$$= \sum_{\tilde{m} \in \mathbb{N}-1} \left(\sum_{k \in \mathbb{N}} a_{-\tilde{m}+R_\mu-1+k,j,i} \binom{-\Delta-\tilde{m}+R_\mu+k}{k} (-z_2')^{-\tilde{m}-1}(-z_0')^k(-z_0)^{\tilde{m}+1-k} \right)$$

for $\mu \in \mathbb{R}/\mathbb{Z}$, $j = 0, \ldots, M$ and $i = 0, \ldots, N$. Then in the polydisk, we have

$$\sum_{\tilde{n} \in -\mathbb{N}} \left(\sum_{k \in \mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta+\tilde{n}+R_\mu+1}{k} (-z_2')^{\tilde{n}-k}(-z_0')^k(-z_0)^{-\tilde{n}} \right) \cdot$$

$$\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i$$

$$= \sum_{\tilde{m} \in \mathbb{N}-1} \left(\sum_{k \in \mathbb{N}} a_{-\tilde{m}+R_\mu-1+k,j,i} \binom{-\Delta-\tilde{m}+R_\mu+k}{k} (-z_2')^{-\tilde{m}-1}(-z_0')^k(-z_0)^{\tilde{m}+1-k} \right) \cdot$$

$$\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i \tag{9.180}$$

for $\mu \in \mathbb{R}/\mathbb{Z}$, $j = 0, \ldots, M$ and $i = 0, \ldots, N$. In particular, since when $|(z_2')^{-1}| < r^{-1}$, $((z_2')^{-1}, z_0)$ is in the polydisk, (9.180) holds for such z_2' and $z_0' = z_0$. We know that the left-hand side of (9.180) with $z_0' = z_0$ is analytic in $(z_2')^{-1}$ for $|1 + z_0(z_2')^{-1}| > |z_0(z_2')^{-1}|$ and $1 > |z_0(z_2')^{-1}|$. In particular, the value at $(z_2')^{-1} = z_2^{-1}$ of the left-hand side of (9.180) with $z_0' = z_0$ is determined by analytic extension from its values on the disk $|(z_2')^{-1}| < r^{-1}$. We also know, from (9.171), that the right-hand side of (9.180) with $z_0' = z_0$ is absolutely convergent when $z_2' = z_2$. Thus as a power series in $(z_2')^{-1}$, the right-hand side of (9.180) with $z_0' = z_0$ is absolutely convergent when $|(z_2')^{-1}| \leq |z_2^{-1}|$. Hence the sum of the right-hand side of (9.180) with $z_0' = z_0$ is analytic in $(z_2')^{-1}$ for $|(z_2')^{-1}| < |z_2^{-1}|$ and is continuous on the closed disk $|(z_2')^{-1}| \leq |z_2^{-1}|$. In particular, the value at $(z_2')^{-1} = z_2^{-1}$ of the right-hand side of (9.180) with $z_0' = z_0$ is determined by analytically extending its values on the disk $|(z_2')^{-1}| < r^{-1}$ to the open disk $|(z_2')^{-1}| < |z_2^{-1}|$ and then taking the limit $(z_2')^{-1} \to z_2^{-1}$. Since (9.180) holds in the disk $|(z_2')^{-1}| < r^{-1}$ and the closed segment from $(z_2')^{-1} = 0$ to $(z_2')^{-1} = z_2^{-1}$ lies in the domain of the function given by the left-hand side of (9.180) (with $z_0' = z_0$), the two sides of (9.180) must be equal when $(z_2')^{-1} = z_2^{-1}$, that is,

$$\sum_{\tilde{n} \in -\mathbb{N}} \left(\sum_{k \in \mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta+\tilde{n}+R_\mu+1}{k} (-z_2)^{\tilde{n}-k}(-z_0)^{-\tilde{n}+k} \right) \cdot$$

$$\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i$$

$$= \sum_{\tilde{m} \in \mathbb{N}-1} \left(\sum_{k \in \mathbb{N}} a_{-\tilde{m}+R_\mu-1+k,j,i} \binom{-\Delta - \tilde{m} + R_\mu + k}{k} (-z_2)^{-\tilde{m}-1}(-z_0)^{\tilde{m}+1} \right) \cdot$$

$$\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i \tag{9.181}$$

for $\mu \in \mathbb{R}/\mathbb{Z}$, $j = 0, \ldots, M$ and $i = 0, \ldots, N$. Hence the right-hand side of (9.172) is equal to

$$\sum_{\mu \in \mathbb{R}/\mathbb{Z}} \sum_{j=0}^{M} \sum_{i=0}^{N} \sum_{\tilde{n} \in -\mathbb{N}} \left(\sum_{k \in \mathbb{N}} a_{\tilde{n}+R_\mu,j,i} \binom{-\Delta + \tilde{n} + R_\mu + 1}{k} e^{(-\Delta+\tilde{n}+R_\mu-k+1)l_{\tilde{p}}(-z_2)} e^{(-\tilde{n}-R_\mu-1+k)l_q(-z_0)} \right) \cdot$$

$$\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i$$

$$= \sum_{\mu \in \mathbb{R}/\mathbb{Z}} \sum_{j=0}^{M} \sum_{i=0}^{N} \left(\sum_{\tilde{m} \in \mathbb{N}-1} \left(\sum_{k \in \mathbb{N}} a_{-\tilde{m}+R_\mu-1+k,j,i} \binom{-\Delta - \tilde{m} + R_\mu + k}{k} \cdot \right. \right.$$

$$\left. \left. \cdot e^{(-\Delta-\tilde{m}+R_\mu)l_{\tilde{p}}(-z_2)} e^{(\tilde{m}-R_\mu)l_q(-z_0)} \right) \right) \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i$$

$$= \sum_{m \in \mathbb{R}} \sum_{j=0}^{M} \sum_{i=0}^{N} \left(\sum_{k \in \mathbb{N}} a_{-m-1+k,j,i} \binom{-\Delta - m + k}{k} e^{(-\Delta-m)l_{\tilde{p}}(-z_2)} e^{ml_q(-z_0)} \right) \cdot$$

$$\cdot \left(l_{\tilde{p}}(-z_2) + \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l-1}}{l} \frac{(-z_0)^l}{(-z_2)^l} \right)^j l_q(-z_0)^i. \tag{9.182}$$

Since the right-hand side of (9.182) is exactly (9.171), which in turn is equal to the right-hand side of (9.170), and the left-hand side of (9.182) is equal to the right-hand side of (9.168), (9.168) holds.

For (9.169), we have the following analogue of (9.170), using (7.9)–(7.10):

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, \, x_2=z_2}$$

$$= \langle e^{z_1 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(\Omega_0(\Omega_{-1}(\mathcal{Y}_2))(w_{(2)}, x_2)w_{(3)}, x_1)w_{(1)}\rangle_{W_4} \Big|_{x_1=e^{\pi i}z_1, \, x_2=z_2}$$

$$= \langle e^{z_1 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(e^{x_2 L(-1)}\Omega_{-1}(\mathcal{Y}_2)(w_{(3)}, e^{\pi i}x_2)w_{(2)}, x_1)w_{(1)}\rangle_{W_4} \Big|_{x_1=e^{\pi i}z_1, \, x_2=z_2}$$

$$= \langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(\Omega_{-1}(\mathcal{Y}_2)(w_{(3)}, e^{\pi i} x_2) w_{(2)}, x_1 + x_2) w_{(1)} \rangle_{W_4} \Big|_{x_1 = e^{\pi i} z_1, \, x_2 = z_2} .$$
(9.183)

This exhibits the format of (9.169), and arguments similar to those in the proof of (9.168) above prove (9.169). $\quad\square$

In the following consequence of Lemma 9.25, we assert that two conditions are equivalent; note that the appropriate hypotheses about the generalized modules and the complex numbers are part of the conditions:

Theorem 9.26 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds. Then the following two conditions are equivalent:*

1. *For any objects W_1, W_2, W_3, W_4 and M_1 of $\mathcal{C}$, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_0| > 0$, and any logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, there exist an object M_2 of $\mathcal{C}$ and logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ and $\mathcal{Y}^2$ of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, such that*

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1 = z_1, \, x_2 = z_2}$$
$$= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_0 = z_0, \, x_2 = z_2}$$
(9.184)

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. (Here the substitution notation is as indicated in (7.13) and (7.14)).

2. *For any objects W_1, W_2, W_3, W_4 and M_2 of $\mathcal{C}$, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_0| > 0$, and any logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ and $\mathcal{Y}^2$ of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, there exist an object M_1 of $\mathcal{C}$ and logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, such that*

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_0 = z_0, \, x_2 = z_2}$$
$$= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1 = z_1, \, x_2 = z_2}$$
(9.185)

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

Proof First we note that if there exist M_2, $\mathcal{Y}^1$ and $\mathcal{Y}^2$, or M_1, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ such that Condition 1 or Condition 2, respectively, holds for some particular $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| >$

$|z_0| > 0$ and for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, then with the same M_2, $\mathcal{Y}^1$ and $\mathcal{Y}^2$, or M_1, $\mathcal{Y}_1$ and $\mathcal{Y}_2$, Condition 1 or Condition 2, respectively, holds for all $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_0| > 0$ and for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. This follows from the analyticity (Proposition 7.14), the $L(-1)$-derivative property for logarithmic intertwining operators, and the general fact that if two analytic functions and their derivatives are equal at a particular point, then they are equal on the intersection of their domains by Taylor's theorem and analytic extension, assuming the intersection is connected. In fact, if Condition 1 holds for some particular $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_0| > 0$ and for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, then by the $L(-1)$-derivative property and the $L(-1)$-bracket relation,

$$
\left(\frac{\partial^k}{\partial(z'_1)^k} \frac{\partial^l}{\partial(z'_2)^l} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1 = z'_1, x_2 = z'_2} \right) \Big|_{z'_1 = z_1, z'_2 = z_2}
$$
$$
= \left(\frac{\partial^k}{\partial(z'_1)^k} \frac{\partial^l}{\partial(z'_2)^l} \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_0 = z'_0, x_2 = z'_2} \right) \Big|_{z'_1 = z_1, z'_2 = z_2}
$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, where z'_1, z'_2 are complex variables and $z'_0 = z'_1 - z'_2$; if $z' = z'_1$, z'_2 or z'_0 is a positive real number, then we compute the derivatives using the branches with $\arg z' \geq 0$. Then by Taylor's theorem, there is an open subset of the region $|z'_1| > |z'_2| > |z'_0| > 0$ whose closure contains (z_1, z_2) such that on the closure of this open subset,

$$
\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_1 = z'_1, x_2 = z'_2}
$$
$$
= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle \Big|_{x_0 = z'_0, x_2 = z'_2} \tag{9.186}
$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. The two sides of (9.186) are analytic in z'_1 and z'_2 on the regions $|z'_1| > |z'_2| > 0$, $\arg z'_1, \arg z'_2 > 0$, and $|z'_2| > |z'_0| > 0$, $\arg z'_2, \arg z'_0 > 0$, respectively, and thus are equal on their intersection, $|z'_1| > |z'_2| > |z'_0| > 0$, $\arg z'_1, \arg z'_2, \arg z'_0 > 0$. Hence (9.186) holds on the region $|z'_1| > |z'_2| > |z'_0| > 0$. The argument for Condition 2 is similar.

We shall prove only that Condition 1 implies Condition 2, the other direction being similar.

Suppose that Condition 1 holds. Then for z_1 and z_2 as in the statement of Condition 2, by the first part of Lemma 9.25, there exist $p, q \in \mathbb{Z}$ such that for any logarithmic intertwining operators $\mathcal{Y}^1$ and $\mathcal{Y}^2$ as in the statement of Condition 2,

$$
\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0 = z_0, x_2 = z_2}
$$
$$
= \langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, y_1) \Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, y_2) \cdot
$$
$$
\cdot w_{(1)} \rangle_{W_4} \Big|_{y_1^n = e^{nl_p(-z_1)}, \log y_1 = l_p(-z_1), y_2^n = e^{nl_q(-z_0)}, \log y_2 = l_q(-z_0)}
$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. The same argument as in the proof of (9.186) above gives

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z'_0, \, x_2=z'_2}$$
$$= \langle e^{z_1 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, y_1)\Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, y_2) \cdot$$
$$\cdot w_{(1)}\rangle_{W_4}\Big|_{y_1^n=e^{nl_p(-z'_1)}, \, \log y_1=l_p(-z'_1), \, y_2^n=e^{nl_q(-z'_0)}, \, \log y_2=l_q(-z'_0)} \tag{9.187}$$

for $|z'_1| > |z'_0| > 0$, $|z'_2| > |z'_0| > 0$ and for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

By Remark 3.28, there exist logarithmic intertwining operators $\tilde{\mathcal{Y}}^1$ and $\tilde{\mathcal{Y}}^2$ of types $\binom{W_4}{W_3 M_2}$ and $\binom{M_2}{W_2 W_1}$, respectively, such that

$$\langle e^{z_1 L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}^1)(w_{(3)}, y_1)\Omega_{-1}(\mathcal{Y}^2)(w_{(2)}, y_2) \cdot$$
$$\cdot w_{(1)}\rangle_{W_4}\Big|_{y_1^n=e^{nl_p(-z'_1)}, \, \log y_1=l_p(-z'_1), \, y_2^n=e^{nl_q(-z'_0)}, \, \log y_2=l_q(-z'_0)}$$
$$= \langle e^{z_1 L'(1)}w'_{(4)}, \tilde{\mathcal{Y}}^1(w_{(3)}, y_1)\tilde{\mathcal{Y}}^2(w_{(2)}, y_2)w_{(1)}\rangle_{W_4}\Big|_{y_1=-z'_1, \, y_2=-z'_0} \tag{9.188}$$

for $|z'_1| > |z'_0| > 0$ and for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. Since the last expression is of the same form as the left-hand side of (9.186), we have from Condition 1 and (9.186) that there exist an object M_3 of $\mathcal{C}$ and logarithmic intertwining operators $\tilde{\mathcal{Y}}^3$ and $\tilde{\mathcal{Y}}^4$ of types $\binom{W_4}{M_3 W_1}$ and $\binom{M_3}{W_3 W_2}$, respectively, such that

$$\langle e^{z_1 L'(1)}w'_{(4)}, \tilde{\mathcal{Y}}^1(w_{(3)}, y_1)\tilde{\mathcal{Y}}^2(w_{(2)}, y_2)w_{(1)}\rangle_{W_4}\Big|_{y_1=-z'_1, \, y_2=-z'_0}$$
$$= \langle e^{z_1 L'(1)}w'_{(4)}, \tilde{\mathcal{Y}}^3(\tilde{\mathcal{Y}}^4(w_{(3)}, y_0)w_{(2)}, y_2)w_{(1)}\rangle\Big|_{y_0=-z'_2, \, y_2=-z'_0} \tag{9.189}$$

for $|z'_1| > |z'_0| > |z'_2| > 0$ and for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. (Note that the inequality $|z'_0| > |z'_2| > 0$ fails for $z'_0 = z_0$ and $z'_2 = z_2$.) Again by Remark 3.28, for any $\tilde{p}, \tilde{q} \in \mathbb{Z}$, there exist logarithmic intertwining operators $\mathcal{Y}^3$ and $\mathcal{Y}^4$ of types $\binom{W_4}{M_3 W_1}$ and $\binom{M_3}{W_3 W_2}$, respectively, such that

$$\langle e^{z_1 L'(1)}w'_{(4)}, \tilde{\mathcal{Y}}^3(\tilde{\mathcal{Y}}^4(w_{(3)}, y_0)w_{(2)}, y_2)w_{(1)}\rangle\Big|_{y_0=-z'_2, \, y_2=-z'_0}$$
$$= \langle e^{z_1 L'(1)}w'_{(4)}, \mathcal{Y}^3(\mathcal{Y}^4(w_{(3)}, y_0)w_{(2)}, y_2) \cdot$$
$$\cdot w_{(1)}\rangle\Big|_{y_0^n=e^{nl_{\tilde{p}}(-z'_2)}, \, \log y_0=l_{\tilde{p}}(-z'_2) \, y_2^n=e^{nl_{\tilde{q}}(-z'_0)}, \, \log y_2=l_{\tilde{q}}(-z'_0)} \tag{9.190}$$

for $|z'_0| > |z'_2| > 0$ and for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. Let z_2^0 be a fixed complex number satisfying $|z_1| > |z_2^0| > 0$ and $|z_0^0| > |z_2^0| > 0$, with $z_0^0 = z_1 - z_2^0$. By using the fact that $\mathcal{Y} = \Omega_{-1}(\Omega_0(\mathcal{Y}))$ and comparing the last expression to the right-hand side of

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(9.169), we see from the second part of Lemma 9.25 that there exist $\tilde{p}, \tilde{q} \in \mathbb{Z}$ independent of $\mathcal{Y}^3$ and $\mathcal{Y}^4$ such that

$$\langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}^3(\mathcal{Y}^4(w_{(3)}, y_0)w_{(2)}, y_2) \cdot$$

$$\cdot w_{(1)} \rangle \Big|_{y_0^n = e^{nl_{\tilde{p}}(-z_2^0)}, \, \log y_0 = l_{\tilde{p}}(-z_2^0) \, y_2^n = e^{nl_{\tilde{q}}(-z_0^0)}, \, \log y_2 = l_{\tilde{q}}(-z_0^0)}$$

$$= \langle w'_{(4)}, \Omega_0(\mathcal{Y}^3)(w_{(1)}, x_1)\Omega_0(\mathcal{Y}^4)(w_{(2)}, x_2)w_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1, \, x_2 = z_2^0}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. The same argument as in the proof of (9.186) gives

$$\langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}^3(\mathcal{Y}^4(w_{(3)}, y_0)w_{(2)}, y_2) \cdot$$

$$\cdot w_{(1)} \rangle \Big|_{y_0^n = e^{nl_{\tilde{p}}(-z'_2)}, \, \log y_0 = l_{\tilde{p}}(-z'_2) \, y_2^n = e^{nl_{\tilde{q}}(-z'_0)}, \, \log y_2 = l_{\tilde{q}}(-z'_0)}$$

$$= \langle w'_{(4)}, \Omega_0(\mathcal{Y}^3)(w_{(1)}, x_1)\Omega_0(\mathcal{Y}^4)(w_{(2)}, x_2)w_{(3)} \rangle_{W_4} \Big|_{x_1 = z'_1, \, x_2 = z'_2} \tag{9.191}$$

for $|z'_1| > |z'_2| > 0$, $|z'_0| > |z'_2| > 0$ and for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. The right-hand side is of course defined for $|z'_1| > |z'_2| > 0$.

By Proposition 7.14, we have that both sides of (9.187), (9.188), (9.189), (9.190) and (9.191) define analytic functions of z'_1 and z'_2 in the indicated regions, with the cuts handled as in Propositon 7.14. Thus when restricted to the region $|z'_1| > |z'_2| > |z'_0| > 0$, the left-hand side of (9.187) and the right-hand side of (9.191) are analytic extensions of each other along loops, avoiding crossing the cuts, starting in the region $|z'_1| > |z'_2| > |z'_0| > 0$, passing through the region $|z'_1| > |z'_0| > 0$, the region $|z'_1| > |z'_0| > |z'_2| > 0$, the region $|z'_0| > |z'_2| > 0$, the region $|z'_1| > |z'_2| > 0$, and coming back to the region $|z'_1| > |z'_2| > |z'_0| > 0$ again. We take $z_1, z_2, z_2^0 \in \mathbb{R}_+$ satisfying $z_1 > z_2 > z_0 > 0$ and $z_1 > z_0^0 = z_1 - z_2^0 > z_2^0 > 0$ and consider the path γ from (z_1, z_2) to (z_1, z_2^0) given by $\gamma(t) = (z_1, (1 - t)z_2 + tz_2^0)$ for $t \in [0, 1]$. Then the product $\gamma^{-1} \circ \gamma$ is a loop starting at the point (z_1, z_2) in the region $|z'_1| > |z'_2| > |z'_0| > 0$, passing through the region $|z'_1| > |z'_0| > 0$, the region $|z'_1| > |z'_0| > |z'_2| > 0$, the region $|z'_0| > |z'_2| > 0$, the region $|z'_1| > |z'_2| > 0$, and coming back to the same point (z_1, z_2) in the region $|z'_1| > |z'_2| > |z'_0| > 0$ again. Thus the value of the right-hand side of (9.191) at (z_1, z_2) is the analytic extension of the left-hand side of (9.187) along the loop $\gamma^{-1} \circ \gamma$, which is homotopic to the trivial loop, and so the analytic extension must give the same value. Thus if we take $\mathcal{Y}_1$ and $\mathcal{Y}_2$ to be $\Omega_0(\mathcal{Y}^3)$ and $\Omega_0(\mathcal{Y}^4)$, respectively, (9.185) holds at this particular point (z_1, z_2). By the discussion in the beginning of this proof, (9.185) holds for all z_1, z_2 satisfying $|z_1| > |z_2| > |z_0| > 0$ and hence Condition 2 holds.

In the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$, the same arguments still hold except that all the logarithmic intertwining operators involved are ordinary intertwining operators. $\quad\square$

Using Theorem 9.26, we now prove that under the global assumptions in Theorem 9.23, the assumptions in Part 1 of Theorem 9.23 (or equivalently, of its reformulation, Corollary 9.24) and the assumptions in Part 2 of Theorem 9.23 (or of Corollary 9.24) are equivalent.

These sets of assumptions, which are stated as Conditions 1 and 2 in the theorem below, are the two (equivalent) statements of what we will call the expansion condition below. From Proposition 4.21, the statement that $\mathcal{C}$ is closed under the $P(z)$-tensor product operation for *some* $z \in \mathbb{C}^\times$ is equivalent to the statement that $\mathcal{C}$ is closed under the $P(z)$-tensor product operation for *every* $z \in \mathbb{C}^\times$. In particular, in the following results, instead of assuming that $W_1 \boxtimes_{P(z_0)} W_2$ and $W_2 \boxtimes_{P(z_2)} W_3$ exist in $\mathcal{C}$ for all objects W_1, W_2 and W_3 of $\mathcal{C}$, we assume that $\mathcal{C}$ is closed under the $P(z)$-tensor product operation for some $z \in \mathbb{C}^\times$. Since Conditions 1 and 2 below are about to be used as the two equivalent formulations of the expansion condition, we include the hypotheses on the generalized modules and the complex numbers in Conditions 1 and 2 (as we did in Theorem 9.26).

Theorem 9.27 *Assume that $\mathcal{C}$ is closed under images and under the $P(z)$-tensor product operation for some $z \in \mathbb{C}^\times$, and that the convergence condition for intertwining maps in $\mathcal{C}$ holds. Then the following two conditions are equivalent:*

1. *For any objects W_1, W_2, W_3, W_4 and M_1 of $\mathcal{C}$, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_0| > 0$, any $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 M_1}$ and $P(z_2)$-intertwining map I_2 of type $\binom{M_1}{W_2 W_3}$, and any $w'_{(4)} \in W'_4$,*

$$(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(2)}(z_0)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). Moreover, for any $w_{(3)} \in W_3$ and $n \in \mathbb{R}$, the smallest doubly graded subspace of $W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}$ containing the term $\lambda_n^{(2)}$ of the (unique) series $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ weakly absolutely convergent to $\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}$ as indicated in the $P^{(2)}(z_0)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_0)$-grading condition) and stable under the action of V and of $\mathfrak{sl}(2)$ (which is a generalized V-module (or a V-module) by Theorem 9.17) is a generalized V-submodule (or a V-submodule) of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$.

2. *For any objects W_1, W_2, W_3, W_4 and M_2 of $\mathcal{C}$, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_0| > 0$, any $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{M_2 W_3}$ and $P(z_0)$-intertwining map I^2 of type $\binom{M_2}{W_1 W_2}$, and any $w'_{(4)} \in W'_4$,*

$$(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). Moreover, for any $w_{(1)} \in W_1$ and $n \in \mathbb{R}$, the smallest doubly graded subspace of $W^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}}$ containing the term $\lambda_n^{(1)}$ of the (unique) series $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ weakly absolutely convergent to $\mu^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}}$ as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition) and stable under the action of V and of $\mathfrak{sl}(2)$

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(which is a generalized V-module (or a V-module) by Theorem 9.17) is a generalized V-submodule (or a V-submodule) of some object of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$.

Proof By Propositions 4.8 and 9.13, together with Proposition 9.8, Condition 1 (respectively, Condition 2) in Theorem 9.26 implies Condition 1 (respectively, Condition 2) in the present theorem. Conversely, by Proposition 4.8 and Theorem 9.23 (also recall the formulation in Corollary 9.24), Condition 1 (respectively, Condition 2) in the present theorem implies Condition 1 (respectively, Condition 2) in Theorem 9.26. Thus the present theorem follows immediately from Theorem 9.26. $\square$

We are finally ready to define formally, in the following precise sense, the main concept whose theory has been developed in this section:

Definition 9.28 Assume that $\mathcal{C}$ is closed under images and under the $P(z)$-tensor product operation for some $z \in \mathbb{C}^{\times}$, and that the convergence condition for intertwining maps in $\mathcal{C}$ holds. We call either of the two equivalent conditions in Theorem 9.27 the *expansion condition for intertwining maps in the category $\mathcal{C}$*.

Then Theorem 9.23 can be reformulated as the following result, stating that the convergence and expansion conditions together with certain "minor" conditions imply both versions of associativity of intertwining maps:

Theorem 9.29 *Assume that $\mathcal{C}$ is closed under images and under the $P(z)$-tensor product operation for some $z \in \mathbb{C}^{\times}$, and that the convergence condition and the expansion condition for intertwining maps in the category $\mathcal{C}$ hold, and assume that*

$$|z_1| > |z_2| > |z_0| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$.

1. *Let I_1 and I_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of types $\binom{W_4}{W_1 \, M_1}$ and $\binom{M_1}{W_2 \, W_3}$, respectively. Then there exists a unique $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2 \; W_3}$ such that*

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle = \langle w'_{(4)}, I^1((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \otimes w_{(3)}) \rangle$$

   for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

2. *Analogously, let I^1 and I^2 be $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{M_2 \, W_3}$ and $\binom{M_2}{W_1 \, W_2}$, respectively. Then there exists a unique $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 \; W_2 \boxtimes_{P(z_2)} W_3}$ such that*

$$\langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}) \rangle = \langle w'_{(4)}, I_1(w_{(1)} \otimes (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \rangle$$

   for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. $\square$

We also have the corresponding reformulation of Corollary 9.24, asserting the associativity of logarithmic and of ordinary intertwining operators, under our global conditions:

Corollary 9.30 *Assume that $\mathcal{C}$ is closed under images and under the $P(z)$-tensor product operation for some $z \in \mathbb{C}^\times$, and that the convergence condition and the expansion condition for intertwining maps in the category $\mathcal{C}$ hold, and assume that*

$$|z_1| > |z_2| > |z_0| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$.

1. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively. Then there exists a unique logarithmic intertwining operator (a unique ordinary intertwining operator in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ of type $\binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2\ W_3}$ such that*

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\ x_2=z_2}$$

$$= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)})\rangle\Big|_{x_0=z_0,\ x_2=z_2}$$

(recalling (4.18), (7.13) and (7.14)) for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. In particular, the product of the logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ and $\mathcal{Y}_2$ evaluated at z_1 and z_2, respectively, can be expressed as an iterate (with the intermediate generalized V-module $W_1 \boxtimes_{P(z_0)} W_2$) of logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) evaluated at z_2 and z_0.

2. *Analogously, let $\mathcal{Y}^1$ and $\mathcal{Y}^2$ be logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Then there exists a unique logarithmic intertwining operator (a unique ordinary intertwining operator in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ of type $\binom{W_4}{W_1\ W_2 \boxtimes_{P(z_2)} W_3}$ such that*

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0=z_0,\ x_2=z_2}$$

$$= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\ x_2=z_2}$$

(again recalling (4.18), (7.13) and (7.14)) for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. In particular, the iterate of the logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ and $\mathcal{Y}^2$ evaluated at z_2 and z_0, respectively, can be expressed as a product (with the intermediate generalized V-module $W_2 \boxtimes_{P(z_2)} W_3$) of logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) evaluated at z_1 and z_2. $\square$

Remark 9.31 Theorem 9.29 or, respectively, Corollary 9.30, in fact says that the product of I_1 and I_2 or, respectively, the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$, uniquely "factors through" $W_1 \boxtimes_{P(z_0)} W_2$, and analogously, that the iterate of I^1 and I^2 or, respectively, the iterate of $\mathcal{Y}^1$ and $\mathcal{Y}^2$, uniquely "factors through" $W_2 \boxtimes_{P(z_2)} W_3$; cf. Remark 8.21.

10 The associativity isomorphisms

We are now in a position to construct our associativity isomorphisms, assuming the convergence and expansion conditions for intertwining maps. The strategy and steps in our construction in this section are essentially the same as those in [H] in the finitely reductive case but in place of the corresponding results in [HL1], [HL2], [HL3] and [H], we have to use virtually all the constructions and results that we have obtained so far in this work. We remark that the construction presented here will make the proofs of the coherence and other properties in our construction of braided tensor category structure straightforward. At the end of this section, we show that the validity of the expansion condition is forced by the assumption of the existence of natural associativity maps, no matter how they may be constructed; this exhibits the naturality of the expansion condition.

In the remainder of this work, in addition to Assumptions 4.1, 5.30 and 7.11, we shall also assume that our category $\mathcal{C}$ is closed under images and that for some $z \in \mathbb{C}^\times$, $\mathcal{C}$ is closed under $P(z)$-tensor products, that is, the $P(z)$-tensor product of $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$ exists (in $\mathcal{C}$). For the reader's convenience, we combine all these assumptions as follows:

Assumption 10.1 *Throughout the remainder of this work, we shall assume the following, unless other assumptions are explicitly made:*

1. *A is an abelian group and $\tilde{A}$ is an abelian group containing A as a subgroup.*

2. *V is a strongly A-graded Möbius or conformal vertex algebra.*

3. *All V-modules and generalized V-modules considered are strongly $\tilde{A}$-graded.*

4. *All intertwining operators and logarithmic intertwining operators considered are grading-compatible.*

5. *$\mathcal{C}$ is a full subcategory of the category $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$ (recall Notation 2.36).*

6. *For any object of $\mathcal{C}$, the (generalized) weights are real numbers and in addition there exists $K \in \mathbb{Z}_+$ such that $(L(0) - L(0)_s)^K = 0$ on the generalized module (when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, the latter assertion holds vacuously).*

7. *$\mathcal{C}$ is closed under images, under the contragredient functor, under taking finite direct sums, and under $P(z)$-tensor products for some $z \in \mathbb{C}^\times$.*

Remark 10.2 From Proposition 4.21, for *every* $z \in \mathbb{C}^\times$, $\mathcal{C}$ is closed under $P(z)$-tensor products. Also, by Proposition 5.37, the assumption that $\mathcal{C}$ is closed under $P(z)$-tensor products for some $z \in \mathbb{C}^\times$ is equivalent to the assumption that for any $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$, $W_1 \boxtimes_{P(z)} W_2$ is an object of $\mathcal{C}$, and in this case,

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'.$$

From now on, for $z \in \mathbb{C}^\times$ we shall take our tensor product bifunctor $\boxtimes_{P(z)}$ to be

$$\boxtimes_{P(z)} = \boxtimes'_{P(z)}.$$

We shall construct our associativity isomorphisms using the next theorem. The proof of this theorem is analogous to the proof of the corresponding statement in Theorem 14.10 in [H], but the results used in the proof below are those developed in the present work from Section 2 through Section 9. We shall be using the usual notation

$$\overline{\eta} : \overline{W_1} \to \overline{W_2}$$

to denote the natural extension of a map $\eta : W_1 \to W_2$ of generalized modules to the formal completions.

We shall be constructing a natural isomorphism between the two functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ in (10.4) below. We first determine how the functor

$$\boxtimes_{P(z_1)} \circ (1 \times \boxtimes_{P(z_2)})$$

acts on maps and elements when the convergence condition holds and when $|z_1| > |z_2| > 0$: Consider maps

$$\sigma_1 : W_1 \to W_4,$$

$$\sigma_2 : W_2 \to W_5,$$

$$\sigma_3 : W_3 \to W_6$$

between objects of $\mathcal{C}$. Recall from Remark 4.25 that for $z \in \mathbb{C}^\times$ the functor $\boxtimes_{P(z)}$ acts on maps and elements by:

$$\overline{\sigma_1 \boxtimes_{P(z)} \sigma_2}(w_{(1)} \boxtimes_{P(z)} w_{(2)}) = \sigma_1(w_{(1)}) \boxtimes_{P(z)} \sigma_2(w_{(2)}), \tag{10.1}$$

and recall that by Proposition 4.23, (10.1) determines the V-module map $\sigma_1 \boxtimes_{P(z)} \sigma_2$ uniquely. We have

$$(\boxtimes_{P(z_1)} \circ (1 \times \boxtimes_{P(z_2)}))(\sigma_1, \sigma_2, \sigma_3) = \boxtimes_{P(z_1)}(\sigma_1, \sigma_2 \boxtimes_{P(z_2)} \sigma_3) = \sigma_1 \boxtimes_{P(z_1)} (\sigma_2 \boxtimes_{P(z_2)} \sigma_3),$$

and the effect of this map on elements is determined as follows: Since

$$\overline{\sigma_2 \boxtimes_{P(z_2)} \sigma_3}(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) = \sigma_2(w_{(2)}) \boxtimes_{P(z_2)} \sigma_3(w_{(3)}),$$

we have (using the projection notation π_n)

$$(\sigma_2 \boxtimes_{P(z_2)} \sigma_3)(\pi_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) = \pi_n(\sigma_2(w_{(2)}) \boxtimes_{P(z_2)} \sigma_3(w_{(3)}))$$

for all $n \in \mathbb{R}$, so that

$$\overline{\sigma_1 \boxtimes_{P(z_1)} (\sigma_2 \boxtimes_{P(z_2)} \sigma_3)}(w_{(1)} \boxtimes_{P(z_1)} \pi_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))$$
$$= \sigma_1(w_{(1)}) \boxtimes_{P(z_1)} \pi_n(\sigma_2(w_{(2)}) \boxtimes_{P(z_2)} \sigma_3(w_{(3)})).$$

Thus for

$$w' \in (W_4 \boxtimes_{P(z_1)} (W_5 \boxtimes_{P(z_2)} W_6))',$$

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we have

$$\langle(\sigma_1 \boxtimes_{P(z_1)} (\sigma_2 \boxtimes_{P(z_2)} \sigma_3))'(w'), w_{(1)} \boxtimes_{P(z_1)} \pi_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)})\rangle$$
$$= \langle w', \sigma_1(w_{(1)}) \boxtimes_{P(z_1)} \pi_n(\sigma_2(w_{(2)}) \boxtimes_{P(z_2)} \sigma_3(w_{(3)}))\rangle,$$

and so by the convergence condition,

$$\langle(\sigma_1 \boxtimes_{P(z_1)} (\sigma_2 \boxtimes_{P(z_2)} \sigma_3))'(w'), w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})\rangle$$
$$= \langle w', \sigma_1(w_{(1)}) \boxtimes_{P(z_1)} (\sigma_2(w_{(2)}) \boxtimes_{P(z_2)} \sigma_3(w_{(3)}))\rangle.$$

Hence

$$\overline{\sigma_1 \boxtimes_{P(z_1)} (\sigma_2 \boxtimes_{P(z_2)} \sigma_3)}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))$$
$$= \sigma_1(w_{(1)}) \boxtimes_{P(z_1)} (\sigma_2(w_{(2)}) \boxtimes_{P(z_2)} \sigma_3(w_{(3)})), \tag{10.2}$$

and by Corollary 7.17, this determines the V-module map $\sigma_1 \boxtimes_{P(z_1)} (\sigma_2 \boxtimes_{P(z_2)} \sigma_3)$ uniquely (cf. (10.1)). Analogously, when the convergence condition holds and when $|z_2| > |z_1 - z_2| > 0$, the functor

$$\boxtimes_{P(z_2)} \circ (\boxtimes_{P(z_1-z_2)} \times 1)$$

acts on elements by:

$$\overline{(\sigma_1 \boxtimes_{P(z_1-z_2)} \sigma_2) \boxtimes_{P(z_2)} \sigma_3}((w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)})$$
$$= (\sigma_1(w_{(1)}) \boxtimes_{P(z_1-z_2)} \sigma(w_{(2)})) \boxtimes_{P(z_2)} \sigma_3(w_{(3)}), \tag{10.3}$$

and by Corollary 7.19, this determines the corresponding V-module map uniquely. The formulas (10.2) and (10.3), which extend (10.1), are crucial.

Theorem 10.3 *Assume that the convergence condition and the expansion condition for intertwining maps in $\mathcal{C}$ (see Definitions 7.4 and 9.28) both hold. Let z_1, z_2 be complex numbers satisfying*

$$|z_1| > |z_2| > |z_1 - z_2| > 0$$

(so that in particular, $z_1 \neq 0$, $z_2 \neq 0$ and $z_1 \neq z_2$). Then there exists a unique natural isomorphism

$$\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : \boxtimes_{P(z_1)} \circ (1 \times \boxtimes_{P(z_2)}) \to \boxtimes_{P(z_2)} \circ (\boxtimes_{P(z_1-z_2)} \times 1) \tag{10.4}$$

such that for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, with W_j objects of $\mathcal{C}$,

$$\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) = (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}, \tag{10.5}$$

where for simplicity we use the same notation $\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}$ to denote the isomorphism of (generalized) modules

$$\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \longrightarrow (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3. \tag{10.6}$$

Proof The uniqueness of $\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}$ follows from Corollary 7.17.

Let $W_{P(z_1,z_2)}$ be the subspace of

$$(W_1 \otimes W_2 \otimes W_3)^*$$

consisting of the elements λ satisfying the following conditions:

1. The $P(z_1,z_2)$-compatibility condition (see Section 8).

2. The $P(z_1,z_2)$-local grading restriction condition (the $L(0)$-semisimple $P(z_1,z_2)$-local grading restriction condition in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) (see Section 8).

3. Either one of the following conditions (see Section 9):

   (a) The $P^{(1)}(z_2)$-local grading restriction condition (the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$).

   (b) The $P^{(2)}(z_1-z_2)$-local grading restriction condition (the $L(0)$-semisimple $P^{(2)}(z_1-z_2)$-local grading restriction condition in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$).

4. Either one of the following conditions, depending on which condition is satisfied in 3 above (that is, either 3(a) and 4(a) hold or 3(b) and 4(b) hold):

   (a) For any $w_{(1)} \in W_1$ and $n \in \mathbb{R}$, the smallest doubly graded subspace of $W^{(1)}_{\lambda,w_{(1)}}$ containing the term $\lambda_n^{(1)}$ of the (unique) series $\sum_{n\in\mathbb{R}} \lambda_n^{(1)}$ weakly absolutely convergent to $\mu^{(1)}_{\lambda,w_{(1)}}$ as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$) and stable under the action of V and of $\mathfrak{sl}(2)$ is a generalized V-module (or a V-module) and is in fact a generalized V-submodule (or a V-submodule) of some object of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$.

   (b) For any $w_{(3)} \in W_3$ and $n \in \mathbb{R}$, the smallest doubly graded subspace of $W^{(2)}_{\lambda,w_{(3)}}$ containing the term $\lambda_n^{(2)}$ of the (unique) series $\sum_{n\in\mathbb{R}} \lambda_n^{(2)}$ weakly absolutely convergent to $\mu^{(2)}_{\lambda,w_{(3)}}$ as indicated in the $P^{(2)}(z_1 - z_2)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-grading condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$) and stable under the action of V and of $\mathfrak{sl}(2)$ is a generalized V-module (or a V-module) and is in fact a generalized V-submodule (or a V-submodule) of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$.

By Proposition 8.5, the natural map

$$\boxtimes_{P(z_1)} \circ (1_{W_1} \otimes \boxtimes_{P(z_2)}) : W_1 \otimes W_2 \otimes W_3 \to \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$$

is a $P(z_1,z_2)$-intertwining map. Recalling Remark 8.12, let

$$\Psi^{(1)}_{P(z_1,z_2)} = (\boxtimes_{P(z_1)} \circ (1_{W_1} \otimes \boxtimes_{P(z_2)}))',$$

the natural map from

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) = (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

to $(W_1 \otimes W_2 \otimes W_3)^*$ given by

$$\begin{aligned}
(\Psi^{(1)}_{P(z_1,z_2)}(\nu))&(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&= \langle \nu, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}
\end{aligned} \qquad (10.7)$$

for

$$\nu \in W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3),$$

$w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Then by Proposition 8.16, $\Psi^{(1)}_{P(z_1,z_2)}$ is an $\tilde{A}$-compatible map and it intertwines the actions of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}]$$

and of $L'_{P(z_1)}(j)$ and $L'_{P(z_1,z_2)}(j)$, $j = -1, 0, 1$, on $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and on $(W_1 \otimes W_2 \otimes W_3)^*$. In particular,

$$\Psi^{(1)}_{P(z_1,z_2)} \circ L'_{P(z_1)}(0) = L'_{P(z_1,z_2)}(0) \circ \Psi^{(1)}_{P(z_1,z_2)}$$

and

$$\Psi^{(1)}_{P(z_1,z_2)} \circ Y'_{P(z_1)}(u, x) = Y'_{P(z_1,z_2)}(u, x) \circ \Psi^{(1)}_{P(z_1,z_2)}$$

for $u \in V$. Thus $\Psi^{(1)}_{P(z_1,z_2)}$ preserves generalized weights, the image

$$\Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))$$

of $\Psi^{(1)}_{P(z_1,z_2)}$ is a generalized module (recall the proof of Proposition 8.17), and $\Psi^{(1)}_{P(z_1,z_2)}$ is a map of generalized modules from

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

to this image.

By Propositions 8.5, 8.17 and 9.13, $\Psi^{(1)}_{P(z_1,z_2)}$ in fact maps $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ into $W_{P(z_1,z_2)}$; the elements of the image satisfy 3(a) and 4(a). By the expansion condition, the elements of this image also satisfy 3(b) and 4(b).

Analogously, let

$$\Psi^{(2)}_{P(z_1,z_2)} = (\boxtimes_{P(z_2)} \circ (\boxtimes_{P(z_1-z_2)} \otimes 1_{W_3}))',$$

the natural map from

$$(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 = ((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)'$$

to $(W_1 \otimes W_2 \otimes W_3)^*$ given by

$$(\Psi^{(2)}_{P(z_1,z_2)}(\xi))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$
$$= \langle \xi, (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle_{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3} \qquad (10.8)$$

for

$$\xi \in (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3, \qquad (10.9)$$

$w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Again by Propositions 8.5 and 8.16, $\Psi^{(2)}_{P(z_1,z_2)}$ is an $\tilde{A}$-compatible map and it intertwines the actions of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}]$$

and of $L'_{P(z_2)}(j)$ and $L'_{P(z_1,z_2)}(j)$, $j = -1, 0, 1$, on $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ and on $(W_1 \otimes W_2 \otimes W_3)^*$, and in particular,

$$\Psi^{(2)}_{P(z_1,z_2)} \circ L'_{P(z_2)}(0) = L'_{P(z_1,z_2)}(0) \circ \Psi^{(2)}_{P(z_1,z_2)}$$

and

$$\Psi^{(2)}_{P(z_1,z_2)} \circ Y'_{P(z_2)}(u, x) = Y'_{P(z_1,z_2)}(u, x) \circ \Psi^{(2)}_{P(z_1,z_2)}$$

for $u \in V$. Thus $\Psi^{(2)}_{P(z_1,z_2)}$ preserves generalized weights, the image

$$\Psi^{(2)}_{P(z_1,z_2)}((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)$$

of $\Psi^{(2)}_{P(z_1,z_2)}$ is a generalized module, and $\Psi^{(2)}_{P(z_1,z_2)}$ is a map of generalized modules from

$$(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

to this image. Again by Propositions 8.5, 8.17 and 9.13, $\Psi^{(2)}_{P(z_1,z_2)}$ maps $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ into $W_{P(z_1,z_2)}$; the elements of the image satisfy 3(b) and 4(b). By the expansion condition, the elements of this image also satisfy 3(a) and 4(a).

We next show that both $\Psi^{(1)}_{P(z_1,z_2)}$ and $\Psi^{(2)}_{P(z_1,z_2)}$ are injective. Let

$$\nu \in W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

be such that

$$\Psi^{(1)}_{P(z_1,z_2)}(\nu) = 0,$$

that is,

$$\langle \nu, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} = 0 \qquad (10.10)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Since $\Psi^{(1)}_{P(z_1,z_2)}$ preserves generalized weights, we can assume that ν is homogeneous. Then (10.10) implies that for all $n \in \mathbb{R}$,

$$\langle \nu, \pi_n(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} = 0$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. But by Corollary 7.17, the elements

$$\pi_n(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))$$

for $n \in \mathbb{R}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ span the space $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$, so that $\nu = 0$, and we have the injectivity of $\Psi^{(1)}_{P(z_1,z_2)}$. The proof of the injectivity of $\Psi^{(2)}_{P(z_1,z_2)}$ is completely analogous.

Now we want to prove that the images of our two maps are equal:

$$\Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)) = \Psi^{(2)}_{P(z_1,z_2)}((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3).$$

Let

$$\lambda \in \Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))$$

and take $\nu \in W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ so that $\lambda = \Psi^{(1)}_{P(z_1,z_2)}(\nu)$. Then by Theorem 9.23, there exists a unique $P(z_2)$-intertwining map I^1 of type

$$\begin{pmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ W_1 \boxtimes_{P(z_1-z_2)} W_2 \qquad W_3 \end{pmatrix}$$

such that

$$\begin{aligned}
\lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) &= (\Psi^{(1)}_{P(z_1,z_2)}(\nu))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&= \langle \nu, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle \\
&= \langle \nu, I^1((w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \otimes w_{(3)}) \rangle \qquad (10.11)
\end{aligned}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$.

By the definition of the $P(z_2)$-tensor product of $W_1 \boxtimes_{P(z_1-z_2)} W_2$ and W_3 in $\mathcal{C}$, there exists a unique map of generalized modules

$$\eta : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

such that

$$I^1 = \overline{\eta} \circ \boxtimes_{P(z_2)}.$$

Then by (10.11), we obtain

$$\begin{aligned}
\lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) &= \langle \nu, I^1((w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \otimes w_{(3)}) \rangle \\
&= \langle \nu, (\overline{\eta} \circ \boxtimes_{P(z_2)})((w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \otimes w_{(3)}) \rangle \\
&= \langle \eta'(\nu), (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle \\
&= (\Psi^{(2)}_{P(z_1,z_2)}(\eta'(\nu)))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})
\end{aligned}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Thus $\lambda = \Psi^{(2)}_{P(z_1,z_2)}(\eta'(\nu))$, proving that λ lies in $\Psi^{(2)}_{P(z_1,z_2)}((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)$. Thus

$$\Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)) \subset \Psi^{(2)}_{P(z_1,z_2)}((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3).$$

The proof of the opposite inclusion,

$$\Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)) \supset \Psi^{(2)}_{P(z_1,z_2)}((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3),$$

is completely analogous.

Since

$$\Psi^{(1)}_{P(z_1,z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to \Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))$$

is injective, we have the natural map

$$(\Psi^{(1)}_{P(z_1,z_2)})^{-1} : \Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)) \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

of generalized modules. Thus we have a natural isomorphism

$$(\Psi^{(1)}_{P(z_1,z_2)})^{-1} \circ \Psi^{(2)}_{P(z_1,z_2)} : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3),$$

of generalized modules. Its contragredient map $\mathcal{A}$ hence gives a natural isomorphism (10.6), and (10.5) indeed holds for the map $\mathcal{A}$. In fact, for ξ as in (10.9), (10.8) holds, but on the other hand, for

$$\nu = (\Psi^{(1)}_{P(z_1,z_2)})^{-1} \Psi^{(2)}_{P(z_1,z_2)}(\xi),$$

(10.7) gives

$$(\Psi^{(2)}_{P(z_1,z_2)})(\xi)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \langle (\Psi^{(1)}_{P(z_1,z_2)})^{-1} \Psi^{(2)}_{P(z_1,z_2)}(\xi), w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle,$$

and this equals

$$\langle \xi, \overline{\mathcal{A}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \rangle.$$

The formulas (10.2), (10.3) and (10.5) exhibit the naturality of (10.4).   □

Definition 10.4 For $z_1, z_2 \in \mathbb{C}$ satisfying

$$|z_1| > |z_2| > |z_1 - z_2| > 0$$

and objects W_1, W_2, and W_3 of $\mathcal{C}$, the *associativity isomorphism from*

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

to

$$(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

is the natural isomorphism $\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}$ given in Theorem 10.6. We also have the natural *inverse associativity isomorphism*

$$\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3). \qquad (10.12)$$

Remark 10.5 The inverse associativity isomorphism $\alpha_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$ satisfies

$$\overline{\alpha_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}((w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}) = w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \qquad (10.13)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, and (10.13) determines (10.12) uniquely.

As in the setting of [H], the existence of such associativity isomorphisms implies the expansion condition and hence that products can be expressed as iterates and vice versa; the following converse of Theorem 10.3 essentially says, then, that the expansion condition is equivalent to the existence of natural associativity isomorphisms:

Theorem 10.6 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ (see Definition 7.4) holds. Suppose that for any complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and any objects W_1, W_2 and W_3 of $\mathcal{C}$, there exists a map $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$ of (generalized) modules of the form (10.6) such that (10.5) holds for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Then $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$ for W_1, W_2 and W_3 is uniquely determined and is a module isomorphism; these maps define a natural isomorphism of functors of the form (10.4); and furthermore, the expansion condition holds, and in particular, products of intertwining maps can be expressed as iterates and conversely, as in Theorem 9.29 and Corollary 9.30. Analogously, suppose that for any complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and for any objects W_1, W_2 and W_3 of $\mathcal{C}$, there exists a map of (generalized) modules of the form (10.12) such that (10.13) holds for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Then the analogous conclusions hold; in particular, the expansion condition again holds, and products of intertwining maps can be expressed as iterates and conversely.*

Proof We shall prove the first half; the second half is proved analogously. We need only prove that given the maps $\mathcal{A}$ as indicated, the expansion condition holds; all the other conclusions are either clear or immediate consequences, using Theorem 10.3 and its proof.

To prove the expansion condition, we shall prove Condition 2 in Theorem 9.27. With z_1 and z_2 as indicated, for any objects W_1, W_2, W_3, W_4 and M_2 of $\mathcal{C}$ and any $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{M_2 W_3}$ and $P(z_1 - z_2)$-intertwining map I^2 of type $\binom{M_2}{W_1 W_2}$, by Proposition 8.19 there exists a unique

$$\widetilde{I}^1 \in \mathcal{M}[P(z_2)]_{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \, W_3}^{W_4}$$

such that

$$I^1 \circ (I^2 \otimes 1_{W_3}) = \widetilde{I}^1 \circ (\boxtimes_{P(z_1-z_2)} \otimes 1_{W_3}).$$

By the definition of $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ (cf. Proposition 4.17), there exists a unique module map

$$\eta : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_4$$

such that

$$\widetilde{I}^1 = \overline{\eta} \circ \boxtimes_{P(z_2)}.$$

Then we have

$$I^1 \circ (I^2 \otimes 1_{W_3}) = \widetilde{I}^1 \circ (\boxtimes_{P(z_1 - z_2)} \otimes 1_{W_3}) = \overline{\eta} \circ (\boxtimes_{P(z_2)} \circ (\boxtimes_{P(z_1 - z_2)} \otimes 1_{W_3})).$$

Thus for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, by (10.5) we have

$$\begin{aligned}
I^1(&I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}) \\
&= \overline{\eta}((w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}) \\
&= \overline{\eta}(\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))). \quad (10.14)
\end{aligned}$$

By Proposition 9.13, for $w' \in W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))$,

$$(\boxtimes_{P(z_1)} \circ (1_{W_1} \otimes \boxtimes_{P(z_2)}))'(w') \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$) and the other condition in Condition 2 in Theorem 9.27. Since η and $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)}$ are module maps, for $w'_{(4)} \in W'_4$ we have that

$$\begin{aligned}
(\overline{\eta} &\circ \overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)}} \circ (\boxtimes_{P(z_1)} \circ (1_{W_1} \otimes \boxtimes_{P(z_2)})))'(w'_{(4)}) \\
&= (\boxtimes_{P(z_1)} \circ (1_{W_1} \otimes \boxtimes_{P(z_2)}))'((\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)})'(\eta'(w'_{(4)})))
\end{aligned}$$

also satisfies Condition 2 in Theorem 9.27. Thus by (10.14), $(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)})$ satisfies this condition, that is, the expansion condition holds. $\square$

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